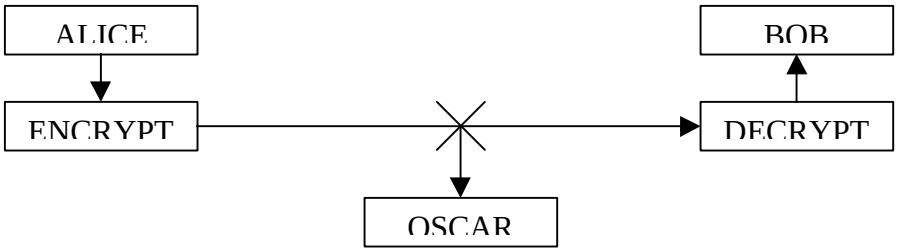
Cryptography

What is cryptography?

- Formal Definition: The art or science concerning the principles, means, and methods for rendering plain information unintelligible, and for restoring encrypted information to intelligible form. (*National Information Systems Security (INFOSEC) Glossary, 1999*)
- Informal: The basic objective of cryptography is to enable two people, whom are usually referred to as Alice and Bob in the text that I read, to communicate over an insecure channel in such a way that an opponent, usually referred to as Oscar, cannot understand what is being said. This channel could be a telephone line or computer network, for example.



Basic Terminology.

- The information that Alice sends to Bob, which is called **plaintext**, can be English text, numerical data, zeros and ones, or anything at all – its structure is completely arbitrary.
- Alice **encrypts** the plaintext, using a predetermined **key**, and sends the resulting **ciphertext** over the channel.
- Oscar, upon seeing the ciphertext in the channel by eavesdropping, cannot determine what the plaintext was.
- But Bob, who knows the **key**, can **decrypt** the ciphertext and reconstruct the plaintext.

This concept leads us to the following definition.

Definition

- A **cryptosystem** is a five-tuple (P, C, K, E, D), where the following conditions are satisfied:
  1. P is a finite set of possible **plaintexts**.
  2. C is a finite set of possible **ciphertexts**.
  3. K, the **keyspace**, is a finite set of possible keys.
  4. For each $k \in K$, there is an **encryption rule** $e_k \in E$ and a corresponding **decryption rule** $d_k \in D$. Each $e_k: P \rightarrow C$ and $d_k: C \rightarrow P$ are functions such that $d_k(e_k(x)) = x$ for all plaintext $x \in P$.

The main property is property 4. It says that if a plaintext x is encrypted using e_k, and the resulting ciphertext is subsequently decrypted using d_k, then the original plaintext x results. Clearly, e_k is one-to-one, otherwise, decryption could not be accomplished in an unambiguous manner. For example, if $y = e_k(x_1) = e_k(x_2)$ where $x_1 \neq x_2$, then Bob has no way of knowing whether y should decrypt to x_1 or x_2.

Two Types of Cryptosystems

- Cryptosystems can be broadly classified into **symmetric-key systems** and **public-key systems**.
- A **public-key system** uses two keys, a public key to encrypt the messages and a private key to decrypt them.
- (say how we will discuss much more about these tomorrow)
- A **symmetric-key system**, or **secret-key system**, in contrast is a cryptosystem in which the sender and the receiver of a message, share a single common key that is used to encrypt and decrypt messages.
- (this is what we will talk about today)

For today, we will be assuming that Alice and Bob have chosen a random key and exchanged secretly either in person or over a secure line. Now lets get to some simple cryptosystems. I will go through this first basic cryptosystem in excrutiating depth in order to really give a feel for what is going on.

Shift Cipher

- The idea behind shift cipher is based on closure and inverses of addition in the group Z_m, where m is a positive integer.
- We will use shift cipher to encrypt ordinary English text by setting up a correspondence between alphabetic characters and residues modulo 26 as follows $A \leftrightarrow 0$, $B \leftrightarrow 1$, …, $Z \leftrightarrow 25$.
- Shift Cipher is defined by $P = C = K = Z_{26}$. For $0 \leq k \leq 25$, define
  $$e_k(x) = (x + k) \text{ (modulo 26)}$$
  and
  $$d_k(y) = (y - k) \text{ (modulo 26)}$$
  for all $x,y \in Z_{26}$.

Is This a Cryptosystem?

- Properties 1, 2, and 3 are clearly met, i.e., the set of plaintexts, the set of ciphertexts, and the keyspace all are finite.
- We must check property 4.
  a) Let $k \in K$ and $x \in P$, arbitrary.
  b) $d_k(e_k(x)) = d_k((x + k)(\text{mod } 26)) = ((x + k) - k) \text{ (mod 26)} = x \text{ (mod 26)} = x$.
- Therefore, Shift Cipher is in fact a cryptosystem.

Here's a simple example of the shift cipher in action. We will refer to the correspondences below for the example.

A	B	C	D	E	F	G	H	I	J	K	L	M
0	1	2	3	4	5	6	7	8	9	10	11	12

N	O	P	Q	R	S	T	U	V	W	X	Y	Z
13	14	15	16	17	18	19	20	21	22	23	24	25

Example

- Suppose the key is k = 11, and the plaintext to be sent is
  marinersarethebest
- We first convert the plaintext to a sequence of integers using the above correspondence, obtaining the following:

12	0	17	8	13	4	17	18	0
17	4	19	7	4	1	4	18	19

- Next, we add 11 to each value modulo 26 to yield:

23	11	2	19	24	15	2	3	11
2	15	4	18	15	12	15	3	4

- Finally, we convert the sequence of integers to alphabetic characters, obtaining the ciphertext:
  XLCTYPCDLCPESPMPDE
- This is the message that is sent. To decrypt the ciphertext, Bob will first convert the ciphertext to a sequence of integers, then subtract 11 from each value (mod 26), and finally convert the sequence of integers to alphabetic characters.

I'm sure that all of us have seen puzzles like this in the newspaper or puzzle books and realize that this is not a difficult system to break. Just trying all the keys, although cumbersome, would not be difficult in the Shift Cipher, that is one of the reasons, in practice, it is not secure. The process of attempting to compute the key k, given a string of ciphertext, is called cryptanalysis. In general, we want cryptosystems in which an exhaustive key search is impossible(even for computers).

Now lets study a cryptosystem that has more algebra involved and a larger keyspace.

Affine Cipher

- In the affine cipher, we restrict our encryption function to functions of the form:
  $e(x) = (ax + b) \pmod{26}$, where $a, b \in Z_{26}$, $a \neq 0$.
- These functions are called **affine functions**, hence the name.
- In order that decryption is possible, it is necessary to ask when an affine function is one-to-one. In other words, for any $y \in Z_{26}$, we want the congruence $ax + b \equiv y \pmod{26}$ to have a unique solution for x.
- This congruence is equivalent to $ax \equiv (y - b) \pmod{26}$
- Now, as y varies over Z_{26}, so, too, does y – b vary over Z_{26}. Hence, it suffices to study the congruence $ax \equiv y \pmod{26}$, for $y \in Z_{26}$, i.e., if this congruence holds for all $y \in Z_{26}$, then it will also hold for all y – b, if $b \in Z_{26}$, because $y - b \in Z_{26}$.

Nonexample: $2x \equiv 0 \pmod{26}$. This has two solution, $x = 0$ and $x = 13$, for the particular $y = 0$ and thus $a = 2$ will not work for a one-to-one affine function.

Trying out some elements of Z_{26}, we start to discover that only some values of a have this property.

Claim: This congruence has a unique solution for every y if and only if $\gcd(a,26) = 1$.
Proof

- '$\Rightarrow$' Suppose the congruence has a unique solution for every y.
  1) (RAA) Assume $\gcd(a,26) \neq 1$.
     Note: a can be 0 because $0x \equiv 0 \pmod{26}$, has 26 solutions.
  2) Then $\gcd(a,26) = d > 1$.
  3) Thus the congruence $ax \equiv 0 \pmod{26}$ has at least two distinct solutions, namely $x = 0$ and $x = 26/d$, for the particular $y = 0$. $x = 0$ works obviously. Because $d \mid a$ and $d \mid 26$, therefore, $\exists m,n \in \mathbf{Z}_{26}$ such that $dm = a$ and $dn = 26$. So $26/d = n$. And $an = (dm)n = (dn)m = 26m \equiv 0 \pmod{26}$.
     This is a contradiction.
  4) Therefore, $\gcd(a,26) = 1$.
- '$\Leftarrow$' Suppose $\gcd(a,26) = 1$.
  1) Assume $x_1, x_2 \in Z_{26}$, such that $ax_1 \equiv ax_2 \pmod{26}$
  2) Then $a(x_1 - x_2) \equiv 0 \pmod{26}$
  3) Thus, $26 \mid a(x_1 - x_2)$.
  4) By Lemma (*), or simply by division property, since $26 \mid a(x_1 - x_2)$, and $\gcd(a,26) = 1$, we must therefore have that $26 \mid (x_1 - x_2)$.
     i.e. $x_1 \equiv x_2 \pmod{26}$ and the congruence has a unique solution.

For example, since $\gcd(4,26) = 2$, it follows that $e_k(x) = 4x + 7$ is not a valid encryption function: x and x + 13 will encrypt to the same value, for any $x \in Z_{26}$. Specifically, $e_k(a) = e_k(0) = 4(0) + 7 \pmod{26} = 7$, and $e_k(n) = e_k(13) = 4(13) + 7 \pmod{26} = 7$.

Lemma(*): If $a,b,c \in Z_{26}$ such that $\gcd(a,b) = 1$ and $a \mid bc$, then $a \mid c$.
Proof

- $a \mid bc \Rightarrow \exists n \in Z_{26}$ such that $an \equiv bc \pmod{26}$.
- $\gcd(a,b) = 1 \Rightarrow \exists r,s \in Z_{26}$ such that $ar + bs \equiv 1 \pmod{26}$.
  (the gcd is a linear combination)
- $(arc + bcs) \equiv c \pmod{26}$ (multiplying by c, distributing, associativity of ring)
- $arc + ans \equiv c \pmod{26}$ (substitute an for bc)
- $a(rc + ns) \equiv c \pmod{26}$ (distribution in Z_{26})
- $rc + ns \pmod{26} \in Z_{26}$, thus $a \mid c$.

Note: There is nothing special about 26, this result can be generalized for any positive integer $m > 1$.

Thm. The congruence $ax \equiv b \pmod{m}$ has a unique solution $x \in Z_m$ if and only if $\gcd(a,m) = 1$.

So we have shown that the only one-to-one affine functions in Z_{26} are those of the form $ax + b \pmod m$ where $\gcd(a,m) = 1$ and $b \in Z_{26}$.

We have shown that the Affine Cipher satisfies the first 3 properties of the definition of a cryptosystem and that it satisfies the need for a one-to-one encryption rule, but we still need an associate decryption rule in order to decrypt the ciphertext.

The Decryption Rule

- Consider our congruence $y \equiv ax + b \pmod{26}$. This is equivalent to
  $$ax \equiv y - b \pmod{26}.$$
  We would like to say that $x \equiv a^{-1}(y - b) \pmod{26}$, but not all $a \in Z_{26}$ have an inverse.
- It turns out that the a values which have a multiplicative inverse modulo 26 are exactly those which satisfy $\gcd(a,26) = 1$.
  (ask if you want to be shown, Lemma (*2))
- Since in encryption the a always satisfies this condition.
- Thus, $x \equiv a^{-1}(y - b) \pmod{26}$, which is the form of the decryption function

Lemma(*2): Let a and n be integers, with $n > 1$. Then a has a multiplicative inverse modulo n if and only if $\gcd(a,n) = 1$.
Proof

- '$\Rightarrow$' First, suppose that a has an inverse modulo n.
  1) Then there is a k such that $ak \equiv 1 \pmod n$.
  2) Thus, n is a divisor of $ak - 1$, so there must be some integer t so that $ak - 1 \equiv nt \pmod n$.
  3) Rewriting this as $ak - nt \equiv 1 \pmod n$, we see that $\gcd(a,n) = 1$.
- '$\Leftarrow$' Now, suppose that $\gcd(a,n) = 1$. Then there are integers r and s with $ar + ns = 1$.
  1) This means $ar - 1$ is divisible by n.
  2) So $ar \equiv 1 \pmod n$.
  3) This tells us r is the inverse of a.

Putting this all together, we finally obtain our Affine Cipher.

Affine Cipher Overview

- The Affine Cipher is defined by $P = C = Z_{26}$ and let
  $$K = \{(a,b) \in Z_{26} \times Z_{26} : \gcd(a,26) = 1\}$$
  For $k = (a,b) \in K$, define
  $$e_k(x) = (ax + b) \pmod{26}$$
  and
  $$d_k(y) = a^{-1}(y - b) \pmod{26}, \text{ for all } x,y \in Z_{26}.$$
- We must check property 4.
  i. Let $k = (a,b) \in K$ and $x \in P$, arbitrary.
  ii. $d_k(e_k(x)) = d_k((ax + b) \pmod{26}) = a^{-1}((ax + b) - b) \pmod{26}$
  $= a^{-1}(ax) \pmod{26} = x \pmod{26}$.
- Therefore, the affine cipher is indeed a cryptosystem.

Example:

- We want to send the word, dog, using the key k = (3,5), gcd(3,26) = 1, so this key is okay.
- We get the corresponding integers as before: 3, 14, 6.
- Then we multiply by 3 and add 5 modulo 26 to yield: 14, 21, 23, so the message is: OVX.
- $3^{-1} = 9$, so to decrypt we take the ciphertext corresponding integers and subtract 5 then multiply by 9 modulo 26 to get back to: 3, 14, 6.

Note: We have developed the set $\{a \in \mathbf{Z}_m : \gcd(a,m) = 1\}$. We will denote this set as $\mathbf{Z}_m^*$. We have introduced that $\mathbf{Z}_m^*$ has inverses, identities, associativity, and commutativity. It turns out that $\mathbf{Z}_m^*$ is in fact an abelian group called the **prime residue group**.

Proof of closure:

Let $x, y \in \mathbf{Z}_m^*$. Then $\gcd(x,m) = 1$ and $\gcd(y,m) = 1$. Then for some $r,s,t,u \in \mathbf{Z}$, $xr + ms \equiv 1 \pmod m$ and $yt + mu \equiv 1 \pmod m$. Multiplying together yields,
$(xr + ms)(yt + mu) \equiv 1 \pmod m$
$xy(rt) + xrmu + ytms + mmsu \equiv 1 \pmod m$,
or $xy(rt) + m(xru + yts + msu) = 1 \pmod m$, $rt \in \mathbf{Z}$, and $xru + yts + msu \in \mathbf{Z}$.
Therefore, $\gcd(xy,m) = 1$. Closure.

Do We Have a Larger Keyspace?

- First a definition, the number of integers in Z_m that are relatively prime to m is often denoted by $\phi(m)$(this function is called the Euler phi-function).
- In the Affine Cipher, the number of choices for b is m, and the number of choices for a is $\phi(m)$. Thus, the number of keys in the keyspace for the Affine Cipher over Z_m is $m\phi(m)$.
- In the special case where m = 26, $\phi(26) = 12$ (1,3,5,7,9,11,15,17,19,21,23,and 25 are all relatively prime to 26), so the size of the keyspace = 26(12) = 312, which is larger, but not secure by any means.

Note: In both the Shift Cipher and the Affine Cipher, once a key is chosen, each alphabetic character is mapped to a unique alphabetic character. For this reason, these cryptosystems are called **monoalphabetic**. These type of systems are easy to break by simply looking for the most common letters in the ciphertext and trying out the most common English letters in their place until an intelligible message is formed.

The following cryptosystem corrects these faults by offering a **polyalphabetic** approach to encryption.

Vigenere Cipher

- The Vigenere Cipher was invented in the sixteenth century, and is associated with the idea of adding a *keyword* of length m to the plaintext that is to be sent.
- This cipher encrypts m alphabetic characters at a time.
- The Vigenere Cipher is defined by $P = C = K = (\mathbf{Z}_{26})^m$.
  For a key $k = (k_1, k_2, \ldots, k_m)$, we define
  $e_k(x_1, x_2, \ldots, x_m) = (x_1 + k_1, x_2 + k_2, \ldots, x_m + k_m)$
  and $d_k(y_1, y_2, \ldots, y_m) = (y_1 - k_1, y_2 - k_2, \ldots, y_m - k_m)$, all operations in $\mathbf{Z}_{26}$.
- Verifications that this is a cryptosystem is left to the reader.

Example:

- Suppose m = 7 and the keyword is ENCRYPT. This corresponds to the numerical key k = (4, 13, 2, 17, 24, 15, 19). And suppose the plaintext is:
  thissystemsecure
- We convert to plaintext and add the key modulo 26, as follows:

19	7	8	18	18	24	18	19	4	12	18	4	2	20	17	4
4	13	2	17	24	15	19	4	13	2	17	24	15	19	4	13
23	20	10	9	16	13	11	23	17	14	9	2	17	13	21	17

- This turns into the ciphertext: XUKJQNLXROJCRNVR
- Decryption is done in the same manner, just subtracting the keyword instead of adding.

Larger Keyspace

- The Vigenere Cipher has a keyspace of 26^m, for a keyword of length m.
- For our example the keyspace is $26^7 = 8031810176$.
- This makes it a lot more difficult to do an exhaustive key search.
- Unfortunately, there are other forms of cryptanalysis and the Vigenere Cipher becomes susceptible to many types if the ciphertext is long.

Let's build another polyalphabetic cryptosystem this time using some linear algebra.

The Hill Cipher

- The strategy behind the Hill Cipher is to take m linear combinations of the m alphabetic characters a plaintext, producing m alphabetic characters in a ciphertext.
- The idea is that if we have a plaintext string of length m, $(x_1, x_2, \ldots, x_m)$
  and m linear combinations: $k_{11}x_1 + k_{12}x_2 + \ldots + k_{1m}x_m$
  $k_{21}x_1 + k_{22}x_2 + \ldots + k_{2m}x_m$
  …
  $k_{m1}x_1 + k_{m2}x_2 + \ldots + k_{mm}x_m$
  Then we can get $(y_1, y_2, \ldots, y_m)$ by:

$$(y_1 \quad y_2 \quad \Lambda \quad y_m) = (x_1 \quad x_2 \quad \Lambda \quad x_m)\begin{pmatrix} k_{11} & k_{12} & \Lambda & k_{1m} \\ k_{21} & k_{22} & \Lambda & k_{2m} \\ \text{M} & \text{M} & \text{O} & \text{M} \\ k_{m1} & k_{m2} & \Lambda & k_{mm} \end{pmatrix} \bmod 26$$

, where $k_{ij} \in \mathbf{Z}_{26}$.

Decryption of the Hill Cipher

- This system seems fine until we consider decryption.
- In order, to retrieve the plaintext from a ciphertext we are going to need the inverse of the encryption matrix, i.e., if $\mathbf{y} = A\mathbf{x}$, then we want to find A^{-1} such that $\mathbf{x} = A^{-1}\mathbf{y}$.
- If all the entries are in a field then we know that the invertible matrices are exactly the ones in the general linear group, however, $\mathbf{Z}_{26}$ is not a field, so what do we do.
- From linear algebra, we know that a real matrix K has an inverse if and only if its determinant is non-zero. We need to find some sort of analygous idea in $\mathbf{Z}_{26}$.

Claim: A matrix K has an inverse modulo 26 if and only if gcd(det K, 26) = 1.
Proof

- '$\Rightarrow$' First, suppose that gcd(det K, 26) = 1.
  1) Then the det K has an inverse in $\mathbf{Z}_{26}$. (We proved this earlier)
  2) Now, for $1 \le i \le m$, $1 \le j \le m$, define K_{ij} to be the matrix obtained from K by deleting the ith row and the jth column.
  3) Define a matrix K* to have as its (i,j)-entry the value $(-1)^{i+j}\det K_{ji}$.
  4) Then it can be shown that $K^{-1} = (\det K)^{-1}K^*$. (skipping over stuff here)
  5) Hence, K has an inverse.
- '$\Leftarrow$' Now, suppose that K has an inverse, K^{-1}.
  1) $1 = \det I = \det (KK^{-1}) = \det K \det K^{-1}$
  2) Thus, det K is invertible in $\mathbf{Z}_{26}$.
  3) Therefore, gcd(det K, 26) = 1.

Hill Cipher Overview

- The Hill Cipher is defined by, for some fixed positive integer m, $P = C = (\mathbf{Z}_{26})^m$, and let K = { A : A is an m x m matrix and gcd(det A, 26) = 1}
  For a key A, we define
  $$e_A(x) = Ax \pmod{26}$$
  and $$d_A(y) = A^{-1}y \pmod{26}, \text{ for all } x,y \in \mathbf{Z}_{26}.$$
- Verifications that this is a cryptosystem is left to the reader.

Example:

- Let m = 2 and take $A = \begin{pmatrix} 2 & 9 \\ 3 & 5 \end{pmatrix}$, to encrypt the plaintext: hello
- First we break up the word into pairs, adding a dummy character, say 'r' in the case of an odd number of characters. This yields: he ll or.
- Converting to integers we get: (7 4) (11 11) (14 17)
- A(7 4) mod 26 = (0 5), A(11 11) mod 26 = (3 24), A(14 17) mod 26 = (1 3)
- The associated ciphertext is: AFDYBD
- det A mod 26 = -17 mod 26 = 9, gcd(9, 26) = 1, so A has an inverse.
- We can get the inverse by taking the normal inverse in **Z** and then taking modulus 26.
- $$A^{-1} = \det(A)^{-1} \begin{pmatrix} 5 & -9 \\ -3 & 2 \end{pmatrix} \bmod 26 = 9^{-1} \begin{pmatrix} 5 & 17 \\ 23 & 2 \end{pmatrix} \bmod 26 = \begin{pmatrix} 3*5 & 3*17 \\ 3*23 & 3*2 \end{pmatrix} \bmod 26$$
  $$= \begin{pmatrix} 15 & 25 \\ 17 & 6 \end{pmatrix}$$
- To recover the plaintext we convert the ciphertext to integer pairs and apply A^{-1}.
- A^{-1}(0 5) mod 26 = (7 4), A^{-1}(3 24) mod 26 = (11 11), A^{-1}(1 3) = (14 17) and we are back to where we started.