

Burnside's Theorem

In order to get into Burnside's Theorem, we need to do some review.

Definition: A **permutation** of a set A is a function from A to A that is both one-to-one and onto. A **permutation group** of a set A is a set of permutations of A that forms a group under function composition.

We will be working with finite groups, however groups of permutations of any non-empty set A of objects do exist.

An example of a permutation on the set is as follows:

Let $A = \{1, 2, 3, 4\}$.

A permutation on A would be $\phi(1) = 4, \phi(2) = 3, \phi(3) = 2, \phi(4) = 1$.

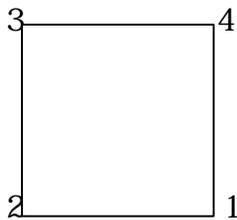
Note: the function is still one-to-one and onto.

A permutation can be written in what is called an array form. This form makes the permutation easier to see.

$$\phi = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

What can we use permutations for?

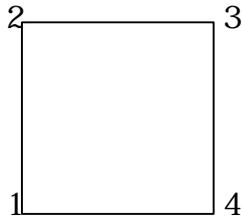
A simple example is to associate motions in D_4 with the permutations of locations of the four corners of the square.



Now suppose we have the permutation

$$\phi = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{bmatrix}$$

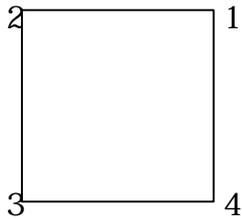
This is a 90-degree rotation (or a rotation by $\Pi/2$) and the result would be



Another permutation is

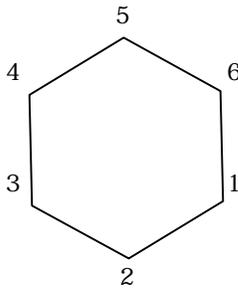
$$\phi = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

which is a reflection across the horizontal axis and it would look like



So, it is easy to see that we can use permutations for symmetries of rotations and reflections.

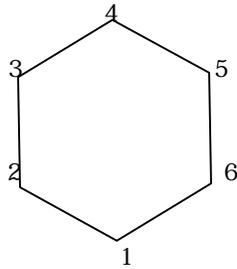
Now, consider the example of a hexagon. A hexagon has six sides and six vertices. We can still do permutations of rotations and reflections. If our original hexagon is labeled as the following:



and we do a 60-degree rotation (or rotate by $\Pi/3$),

$$\phi = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$

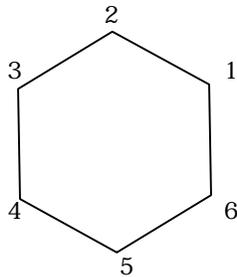
Then our hexagon would look like



A permutation of a reflection across the horizontal axis of our original hexagon would be

$$\phi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$$

And it would look like



Now, if we make three of the vertices white and three of the vertices black (this is what is called a **necklace**), how many different combination possibilities are there?

Since there are 6 vertices with 3 colored the same, the answer is

$$\binom{6}{3} \quad \text{This is read as "six choose three".}$$

This is calculated by . . .

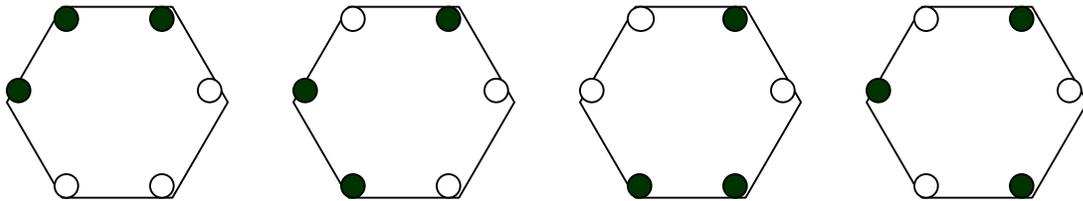
$$\frac{6!}{3!(6-3)!}$$

Which equals 20.

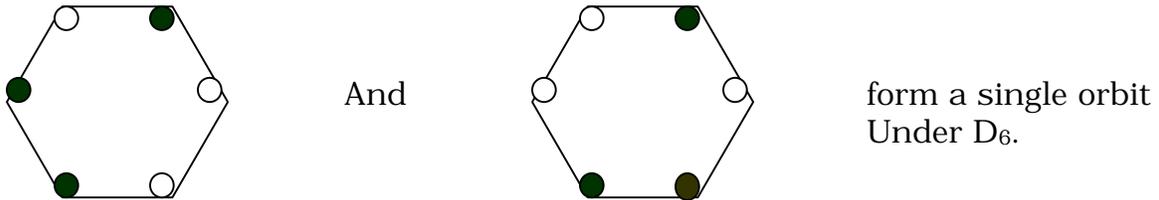
However, if we consider that some of these combinations are **equivalent under the group** of rotations or reflections of a hexagon, then we can say that two arrangements A and B are equivalent under a group G of permutations if there is an element ϕ in G such that $\phi(A) = B$.

The number of unequivalent arrangements under G is the number of **orbits** of G .

Recall the definition of orbit, which was $O_s = \{g(s) : g \in G\}$. Notice there are four orbits under the group of rotations. These orbits consist of rotations of these orbits:



However, there are only three orbits under the group D_6 since



This is where our equivalence under the group definition comes into play by both of these necklaces form a single orbit. Thus, $g(A) = B$. And $g(B) = A$. Thus, we could get all 20 arrangements from 4 different hexagons, but we could get all 20 necklaces from just 3 hexagons.

These examples are simple and we can obtain this information through simple observations. But, what about obtaining information from more complicated examples?

First we need to introduce some notation.

Old Notation:

If G is a group of permutations on a set S and $s \in S$, then
 $G_s = \{g \in G : g(s) = s\}$ (Note: This is our old definition of Stabilizer)
 and $O_s = \{g(s) : g \in G\}$. (Note: This is our old definition of Orbit)

New Notation:

If G is a group of permutations on a set S and $g \in G$, then
 $\text{fix}(g) = \{s \in S : g(s) = s\}$.
 This set is called the **elements fixed by g** .

Burnside's Theorem

If G is a finite group of permutations on a set S and N is the number of orbits of G on S , Then

$$N = \frac{1}{|G|} \sum_{g \in G} |\text{fix}(g)|$$

Proof:

Let the set A be all (s, g) with $s \in S$, $g \in G$, and $g(s) = s$. We count A in two different ways. First, for each $g \in G$, the number of pairs is exactly the order of $\text{fix}(g)$, as s runs over S . So,

$$\# \text{ of pairs of } (s, g) = \sum_{g \in G} |\text{fix}(g)|$$

The second way to count these pairs is for each $s \in S$, the order of G_s is equal to the number of pairs of (s, g) , as g runs over G . Thus,

$$\# \text{ of pairs of } (s, g) = \sum_{s \in S} |G_s|$$

If x_1, x_2, \dots, x_n is a chosen set of representatives where each x_i , (where $i = 1, 2, \dots, n$) is from each orbit of S under G , and x is in the same orbit as x_j , then $O_x = O_{x_j}$ and the number of $G_x = \text{number of } G_{x_j}$.

A theorem used from last semester tells us

$$|G| = |O_x| \cdot |G_x|$$

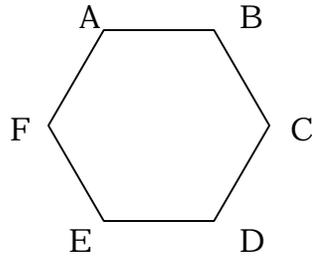
Hence,

$$\sum_{g \in G} |\text{fix}(g)| = \sum_{x \in S} |G_x| = |G| \cdot N$$

(Remember that N is the number of orbits).

Perhaps this can be better shown from a chart.

If we go back to our hexagon example and label the vertices with letters this time, our hexagon would look like:



Our set is $S = \{ A, B, C, D, E, F \}$
 Let our group be $G = \{e, l_h\}$.

Our notation of (s, g) is for all $s \in S$ (like $A \in S$ for this example), and $g \in G$ (like $e \in G$ for this example).

If we put a value of 1 in the (s, g) position if $g(s) = s$ and 0 in the entries otherwise, our chart looks as follows:

	A	B	C	D	E	F	Total	
e		1	1	1	1	1	1	6
l(h)		0	0	1	0	0	1	2
Total		1	1	2	1	1	2	8

Notice $g(s) = s$ is the definition of $\text{fix}(g)$.
 $G_s =$ the sum of the bottom totals. That is,
 $G_s = 1+1+2+1+1+2 = 8$
 Furthermore, the $\text{fix}(g) = 6+2 = 8$.
 Our proof is complete.

Since our proof is complete, we can use Burnside's Theorem.
 If we apply this theorem to our example of the hexagon with our black and white vertices, we can show that the number of orbits is 4. Remember we have 20 arrangements of these vertices by rotations. Thus, our group is now $G = \{e, \rho_{(\pi/3)}, \rho_{(2\pi/3)}, \rho_{\pi}, \rho_{(4\pi/3)}, \rho_{(5\pi/3)}\}$.
 If we make a chart for this, it would look like:

$g \in G$	# of arrangements $\text{fix}(g)$

e	20
$\rho_{(\pi/3)}$	0
$\rho_{(2\pi/3)}$	2
ρ_{π}	0
$\rho_{(4\pi/3)}$	2
$\rho_{(5\pi/3)}$	0

Using Burnside's Theorem:

$$N = 1/6 (20+0+2+0+2+0) = 24/6 = 4$$

It worked and we got the same answer as our observed answer.

Now we will see if Burnside's Theorem will work for counting the number of orbits of our necklaces. We said earlier that the equivalence under the group D_6 gave us 3 different necklaces.

Our new group is $G = \{ e, \rho_{\pi}, \rho_{(2\pi/3)} \text{ or } \rho_{(4\pi/3)}, \rho_{(\pi/3)} \text{ or } \rho_{(5\pi/3)}, \text{ reflect across diagonal, reflect across side bisector} \}$.

Our new chart looks like:

$g \in G$	# of g	# of arrangements fix (g)
e	1	20
ρ_{π}	1	0
$\rho_{(2\pi/3)}$ or $\rho_{(4\pi/3)}$	2	2
$\rho_{(\pi/3)}$ or $\rho_{(5\pi/3)}$	2	0
R diagonal	3	4
R side bisector	3	0

Using Burnside's Theorem we get

$$N = 1/12 (1 \cdot 20 + 1 \cdot 0 + 2 \cdot 2 + 2 \cdot 0 + 3 \cdot 4 + 3 \cdot 0) = 36/12 = 3$$

And we can see that Burnside's theorem gives us the answer we had before for this too.

Who uses Burnside's theorem?

Mathematicians, chemists, and engineers. Chemists use it to count molecules and engineers can use it to count permutations of circuit switches.

Group Action (homomorphism)

So far everything we have done is pretty informal. To make our counts formal:

Let G be a group, S a set of objects and G acts on S if there exists a homomorphism $\phi : G \rightarrow \text{sym}(S)$

Where $\text{sym}(S)$ is the group of all permutations on S .

Let ψ_g denote the image of g under ψ .

If $x, y \in S$, Then $x = y$ iff $\psi_g(x) = y \quad \forall g \in G$.

Note that if ψ is one-to-one, $g \in G$ may be permutations on S . But, if ψ is not one-to-one, elements in G could still be permutations on S . However, that would imply there exists distinct elements $g, h \in G$ such that $\psi_g(x) = \psi_h(x) \quad \forall x \in S$. This means g and h induce the same permutation on S .

Example of group action:

Let S be the 2 diagonals of the square and let G be D_4 (group of symmetries of the square). Then $e = \Psi_{R 0^\circ}, \Psi_{R 180^\circ}, \Psi_D, \Psi_{D'}$

And $\Psi_{R 90^\circ}, \Psi_{R 270^\circ}, \Psi_H, \Psi_V$ switch the diagonals.

Also, $\phi: g \rightarrow \Psi_g$ from D_4 to $\text{sym}(S)$ is a group homomorphism.

Dodecahedron

Question: How many ways is it possible to color a regular dodecahedron so that 5 of its faces are black and the other 7 faces are white?

There are 12 total faces, so for 5 black faces, the answer is

$$\binom{12}{5}$$

$$= \frac{12!}{5!(12-5)!} = 792.$$

There are 3 types of rotations and the identity. The rotations are in the center of a line (R1), where three vertices meet (R2), in the center of a face of the dodecahedron (R3), and the identity (e). We can count the colorings by making another chart:

g in G	Order of g	Number of g, call it s	# fix (g)	(s)(# fix (g))
e	1	1	792	792
R1	2	15	0	0
R2	3(10)(2)=20		0	0
R3	5(6)(4)=24		2	48
Total		60		840

Here 60 is the order of G and 840 is the order of fix (g). The order of fix R2 is 0 because R2 does not fix any face and R2 permutes the faces in disjoint 3 cycles. 5 black faces cannot be permuted by disjoint 3 cycles without fixing 2 faces, thus the order of R2 is 0.

The argument is similar for the order of fix R1. Except R1 is a 2 cycle.

Using Burnside's theorem, we see $N = 840/60 = 14$.

Cycle Index

Suppose g is a rigid motion. To review what a rigid motion is, a rigid motion is rotations and reflections that preserve the original shape of the object. Let K be the number of sets of vertices that can take any color. If k can be any arbitrary colors, and a colors are available, then the order of fix (g) = a^k . This implies that k can take arbitrary colors from the cycle representation of g as a permutation on the set S of vertices.

Example:

If we're using a necklace in D_4 , and one of the points is fixed by a rigid motion, and the other three vertices can be any arbitrary color, then the order of fix (g) = number of colors available to the third power. Since there are 3 disjoint cycles in this example, then there will be three factors of the number of colors in fix (g).

A clean definition of **cycle index**: Suppose that $g \in G$ (where G is a group of permutations). If g breaks down into j_1 cycles of length 1, j_2 cycles of length 2, j_n cycles of length n, then we call (j_1, j_2, \dots, j_n) the **cycle index** of g.

The cycle index is a tool for providing a way to use Burnside's Theorem for counting patterns without having to refer to a list of possible

arrangements. That is the order of $\text{fix}(g) = a^k$ in Burnside's equation. Where k is the number of disjoint cycles.

Definition: **Cycle index polynomial**

$$Z(G) = \frac{1}{|G|} \sum_{x_1^{j_1} x_2^{j_2} \dots x_n^{j_n} \in \mathbf{Q}[x_1, x_2, \dots, x_n]}$$

The proof for this is lengthy, so it won't be done here, but our result for our hexagon example is:

$$f(x_1, x_2, \dots, x_n) = Z(D_6) = (1/12)(x_1^6 + 3x_1^2x_2^2 + 4x_2^3 + 2x_3^2 + 2x_6)$$

So, if only two colors are available for our beads, then our equation looks like:

$$f(2, 2, 2, 2, 2, 2) = Z(D_6) = (1/12)(2^6 + 3(2)^2(2)^2 + 4(2)^3 + 2(2)^2 + 2(2))$$

which equals $156/12 = 13$.

Thus, we have 13 distinct necklaces.

References:

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