Group Representations: **Definition:** If there exists a set T of linear operators $T(G_i)$ in a vector space L which correspond to the elements G_i of a group G such that

$$T(G_a)\,T(G_b) = T(G_a G_b)$$

then this set of operators is said to form a representation of the group G in the space L.
Note: These representations can be of arbitrary dimension. We will always use matrices to denote representations.
Ex. Let $G = \{G_\theta\}$ be the group of counter-clockwise rotations by an angle θ about the z-axis (i.e $G = SU(2)$). Then if we work in the vector space $\mathbf{R}^2$ with unit vectors $e_i = \{\hat{x}, \hat{y}\}$ we can construct the representation which is the set of two-by-two unitary matrices with positive determinant $T(G_\theta) = \langle e_i, G_\theta e_j \rangle = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. Note that $T(G_a)\,T(G_b) = T(G_a G_b)$. If we work in the vector space $\mathbf{R}^3$ with unit vectors $e_i = \{\hat{x}, \hat{y}, \hat{z}\}$ then we can create the representation $\langle e_i, G e_j \rangle = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Generalization to function spaces: **Thm**. Let L be the space of functions $\psi(\vec{r})$ and $G = \{G_i\}$ be the group of coordinate transformations such that if $\psi(\vec{r}) \in L$ then $\psi(G_a^{-1}\vec{r}) \in L$. We can then define a representation T in the function space L such that

$$T(G_a)\,\psi(\vec{r}) = \psi\left(G_a^{-1}\vec{r}\right).$$

Pf. First define $\psi'(\vec{r}) = \psi(G_b \vec{r})$. Note that

$$\begin{aligned} T(G_a)\,T(G_b)\,\psi(\vec{r}) &= T(G_a)\,\psi\left(G_b^{-1}\vec{r}\right) = T(G_a)\,\psi'(\vec{r}) = \psi'\left(G_a^{-1}\vec{r}\right) \\ &= \psi\left(G_b^{-1}G_a^{-1}\vec{r}\right) = \psi\left((G_a G_b)^{-1}\vec{r}\right) = T(G_a G_b)\,\psi(\vec{r}). \end{aligned}$$

Since $T(G_a)\,T(G_b) = T(G_a G_b)$, this is a valid representation.
The matrix representation $T(G_a)$ can be obtained by simply calculating $T(G_a) = \left\langle \psi_i(\vec{r}), G_a \psi_j(\vec{r}) \right\rangle$ where $\psi_i(\vec{r})$ is some basis of the function space L. After we find this matrix, we can calculate $\psi'(\vec{r}) = T(G_a)\,\psi(\vec{r}) = \psi(G_a^{-1}\vec{r})$.
So we see that we can make group representations for spatial variables and, as a more general case, functions. For the remaining of this presentation, when I write e_i as a basis, it represents both a basis for a "normal" space such as $\mathbf{R}^3$ and a basis for a function space.
Definition: If L_1 is a subspace of L such that for all $e_i \in L_1$ it is the case that $T(G_a)\,e_i \in L_1$ then we say that L_1 is invariant with respect to the transformations induced by G, or an invariant subspace.
Ex. Once again let $G = \{G_\theta\}$ be the group of counter-clockwise rotations by an angle θ about the z-axis. We found that if our basis $e_i = \{\hat{x}, \hat{y}, \hat{z}\}$, then a representation of G is the set of matrices $\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$. If we define $L_1 =$span$\{\hat{x}, \hat{y}\}$ and $L_2 =$span$\{\hat{z}\}$ then L_1 and L_2 are invariant subspaces.

Reducibility: Definition: Let L be a space which is invariant with respect to the transformations $T(G_a)$ induced by some group $G = \{G_a\}$. If $L_1, L_2 \subset L$ such that L_1, L_2 are invariant subspaces and L_2 is the orthogonal complement of L_1, then T is reducible with respect to L.
Generally, we can determine whether a matrix representation is reducible by looking at the off diagonal elements in the matrix. If they are zero, then the representation is reducible. If they are not zero, then the representation is not reducible.
Theorem: If $L_1, L_2 \subset L$ such that L_2 is the orthogonal complement of L_1, then L_1 is an invariant subspace wrt a unitary representation T if and only if L_2 is an invariant subspace wrt to T.
Pf: Let the basis of L_1 and L_2 be $\{e_i\}$ and $\{e_j\}$ respectively where the basis vectors can be functions or coordinates. From the invariance of L_1, we have that $\langle T(G_a) e_i, e_j \rangle = 0$ for all G_a. Since T is unitary, the invariance condition on L_1 can be transformed to $\langle e_i, T(G_a^{-1}) e_j \rangle = 0$, which means that L_2 is invariant wrt T.
Note: We will prove later that if $T(G_a)$ is a transformation in the set of representations of a group G, then $T(G_a)$ can be written as a unitary matrix (i.e. $T(G_a)^{-1} = T(G_a)^t$).
As a consequence of this theorem, if we have orthogonal subspaces such that $L = L_1 + L_2 + L_3 + ...$ and each of the L_i's is irreducible with respect to a transformation $T(G_a)$, then we can write the reduction of the representation as

$$T(G_a) = T^{(1)}(G_a) + T^{(2)}(G_a) + T^{(3)}(G_a) + ...$$

where $T^{(i)}(G_a)$ represents the irreducible representation induced in the space L_i (the + sign does not represent normal addition). In this case, the reduced form of $T(G_a)$ can be written in matrix form as

$$\begin{pmatrix} T^{(1)} & & & \\ & T^{(2)} & & \\ & & T^{(3)} & \\ & & & \ddots \end{pmatrix}$$

where each $T^{(i)}$ is an irreducible matrix representation of the group G.
Ex. To create the irreducible matrix representation of $G = \{G_\theta\}$ the group of counter-clockwise rotations by an angle θ about the z-axis in $\mathbf{R}^3$, we simply note that $L_1 =$span$\{\hat{x}, \hat{y}\}$ and $L_2 =$span$\{\hat{z}\}$ are orthogonal complements. If we look at the matrix $\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$, we see that the upper left two-by-two matrix has off-diagonal entries. Therefore the representation is irreducible in the $\{\hat{x}, \hat{y}\}$ and $\{\hat{z}\}$ basis. Therefore we write $L = L_1 + L_2$ and $T = T^{(1)} + T^{(2)} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} + 1$ so that $\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} T^{(1)} & \\ & T^{(2)} \end{pmatrix}$. This is a bad example since the matrix is already in irreducible form, but it gives the general idea.