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Tessellations: The Importance of Symmetry

HISTORY

Although tessellations have been traced back to ancient human cultures and are found in the natural world, they have had a relatively short history as a topic for serious mathematical and scientific study in comparison to the ancient Egyptians. Their mathematical history dates back 1650 BC and even earlier.[1] Johannes Kepler conducted one of the first mathematical studies of tessellations in 1619. He wrote about the regular and semi-regular tessellations (coverings of a plane with regular polygons). Semi-regular tessellations have two properties: (1) it is formed by regular polygons; (2) the arrangement of polygons at every vertex point is identical.[2] A regular tessellation is formed of congruent regular polygons. *Regular* means that the sides of the polygon are all congruent. However, about two hundred years passed before new scientific progress concerning tessellations was made.

Steven Schwartzman's *The Words of Mathematics* gives the definition

> **tessellate** (verb), **tessellation** (noun): from Latin *tessera* a square tablet or a die used for gambling. Latin *tessera* may have been borrowed from Greek *tessares*, meaning four, since a square tile has four sides. The diminute of *tessera* was *tessella*, a small, square piece of stone or a cubical tile used in mosaics. Since a mosaic extends over a given area without leaving any region uncovered, the geometric meaning of the word tessellate is to cover the plane with a pattern in such a way as to leave no region uncovered. By extension, space or hyperspace may also be tessellated.[3]

[1] http://www-history.mcs.st-andrews.ac.uk/history/HistTopics/Egyptian_mathematics
[2] http://mathforum.org/sum95/suzanne/whattess
[3] Steven Schwartzman's *The Words of Mathematics*

COMMON APPLICATIONS

Aside from being studied in mathematical research, tessellations and tilings have been linked with x-ray crystallography and other fields. Tessellations are also sometimes known as tilings, but the word "tilings" usually refers to the patterns of polygons (shapes with straight boundaries), which is a more restrictive category of repeating patterns. X-ray crystallography is a field of science concerned with the repeating arrangements of identical objects as found in nature, a description very similar to the geometrical definition of tessellation. Interestingly, the discoveries made in x-ray crystallography during the mid-20th century are similar to many of the discoveries the Dutch artist M. C. Escher made while formulating designs for his tessellated artwork.[4]

The symmetry issues that are so important in tessellations have been shown to be relevant to quantum mechanics, the study of particles smaller than atoms. These issues include the use of transformations, otherwise known as the techniques of symmetry, and are discussed later. Other fields of research associated with tiling include geology, metallurgy, biology, and cryptology. Figure 1 suggests the relationship between tessellations, symmetry, and x-ray crystallography.

[4] http://library.thinkquest.org/16661/history

FIGURE 1

The original caption read,
"(Left) Electron diffraction patterns of quasicrystalline aluminum-manganese. (Top left) View is along the fivefold symmetry axis; (Center left) rotating by 37.4 reveals the threefold axis, and rotating by 58.3 reveals the twofold axis. (Right) Corresponding views of icosahedrons show that quasicrystalline symmetries match those of the icosahedrons."[5]

SYMMETRY

Symmetry, the preservation of form and configuration across a point, a line, or a plane, is the foundation of every tessellation. Informally, symmetry provides the ability to take a shape and match it exactly to another shape. The techniques are called transformations and include rotations (Figure 4), reflections (a mirror image), glide reflections (Figure 6), and translations (Figure 3). A definition of symmetry, as found in the Random House Dictionary, is as follows, "the correspondence in size, form, and arrangement of parts on opposite sides of the plane, line, or point."[6]

The following figure, Figure 2 shows the eight lines of symmetry of a regular octagon and the five lines of symmetry of a regular pentagon. Along these lines, any transformations can be performed with various polygons such as triangles and rectangles

[5] Figure from http://library.thinkquest.org/16661/history/math
[6] Random House Dictionary, p. 882.

(displayed in Figure 5). However, each of the transformations mentioned above produce a different type of symmetry, and all of them are explained later.[7]

FIGURE 2

An example of translational symmetry is shown in Figure 3. It displays a simple translation (a movement) of a point (left) to form another point (right). For translation, two specifications, direction and magnitude, are needed. Direction can be measured in degrees and magnitude can be measured in inches.[8]

FIGURE 3

Another example of a transformation is rotational symmetry. Rotations turn the shape around a central point. Figure 4 presents a simple rotation of a point (left) to form

[7] Figure from http://library.thinkquest.org/16661/background/symmetry
[8] Figure from http://library.thinkquest.org/16661/background/symmetry

another point (right). Also notice that the distances from the points to the center of rotation remain the same because the shape rotates around a circle.[9]

FIGURE 4

GEOMETRY

Symmetry involves a great deal of geometry such as angle measurements and polygons. The measure of an angle refers to the amount of rotation needed to superimpose one of the line segments onto the other. Most importantly, every time another side is added to a regular polygon, a triangle is in effect added on to the polygon. An example can be explained using two regular polygons, a triangle and a rectangle. Now remember, a regular polygon has congruent interior angles and sides of equal length. A triangle has three sides and an interior angle sum of 180°, and a rectangle has four sides with an interior angle sum of 360°. Another side is added to a triangle to form a rectangle; therefore, adding 180° to the interior angle sum. The same holds for pentagons (5 sides) and hexagons (6 sides). This is shown in Figure 5.

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[9] Figure from http://library.thinkquest.org/16661/background/symmetry

FIGURE 5

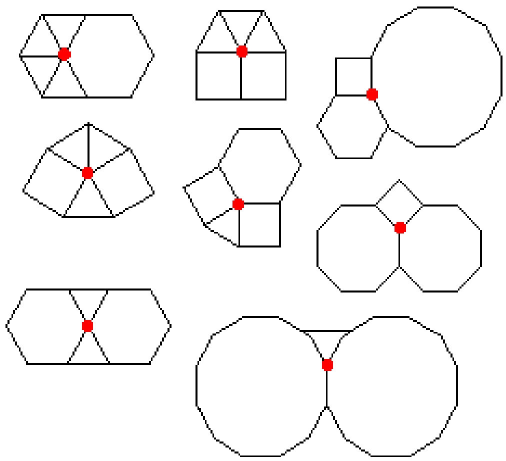
name	number of sides	sum of interior angles	interior angle
triangle	3	180	60
square	4	360	90
pentagon	5	540	108
hexagon	6	720	120
heptagon	7	900	128 4/7
octagon	8	1080	135
nonagon	9	1260	140
decagon	10	1440	144
undecagon	11	1800	150
dodecagon	12	2340	156
...	...	...	...
n-gon	n	180(n-2)	180(n-2)/n

Interior angle measures in regular polygons

In Figure 6, three regular tessellations and eight semi-regular tessellations are shown. Regular polygons are used in all of the following tessellations. Angle measurements play an important role in the symmetry of these examples. First, in the tessellation of equilateral triangles, each triangle has an interior angle sum of 180°. Each angle is 60° and allows the triangles to fit together in a tessellation. Secondly, each angle in the tessellation of squares is a right angle. Squares fit nicely together because all the angles are congruent (90°) and can be used in all of the different transformations. Thirdly, the hexagons form a tessellation because all of the interior angles are all 120°. If one of the interior angles were not 120°, then that hexagon would not tessellate.[10]

[10] Figure from http://mathforum.org/sum95/suzanne/whattess

FIGURE 6



A tessellation of triangles

A tessellation of squares

A tessellation of hexagons

Semi-regular Tessellations *Regular Tessellations*

A reflective image can be seen in the tessellation of triangles. The horizontal line through the middle acts as mirror. The top triangles are exactly the same as the ones on below the line. This is an example of reflective symmetry.

RECENT APPLICATIONS

Symmetry plays a very important role in the formation of tessellations created by M. C. Escher. The third section of his theorem can be proven using rotational and translational symmetry properties of the tessellation of hexagons.[11]

Figure 7 is one of M. C. Escher's tessellations.[12] The symmetry amongst the winged unicorns is amazing, and it is surprising how well they fit together. Looking closely, it is evident that each unicorn is exactly the same; the wings fit between the legs

[11] *Mathematics Magazine*, Volume 64, Number 4, p. 245.
[12] Figure from http://library.thinkquest.org/16661/escher/biography

and the horn fits in the space under the head. This tessellation is an example of glide reflective symmetry which results from the transformation called glide reflection. A glide reflection is actually a combination of a reflection and a translation. The figure that results after a reflection and translation (Figure 7) is simply called the glide reflection of the original figure. The winged unicorns are mirror images but moved up or down slightly to fit into the original figure.

FIGURE 7

An Escher work of art

References

(1) http://library.thinkquest.org

(2) http://mathforum.org/sum95/suzanne/whattess

(3) http://www-history.mcs.st-andrews.ac.uk/history/HistTopics/Egyptian_mathematics

(4) http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Kepler

(5) J. F. Rigby, *Mathematics Magazine*, “Napoleon, Escher, and Tessellations.” Volume 64, Number 4, October 1991. Pages 242-246.

(6) Random House Dictionary, Random House, Inc. New York, New York. Copyright © 1992.

(7) Steven Schwartzman’s *The Words of Mathematics*. The Mathematical Association of America, 1994.