

## Hyperbolic Quadrilaterals

Quadrilaterals are amazing things. They seem like such a simple idea, yet historically they have provided immense amounts of information. It is a commonly known fact that in hyperbolic geometry a rectangle can not exist. This has been verified by proofs that, assuming hyperbolic geometry is consistent, prove that rectangles can not exist in that geometry. But a rectangle is only one form of a quadrilateral, and quadrilaterals can and do exist in other geometries. This paper will examine two special cases of quadrilaterals, the Saccheri Quadrilateral and the Lambert Quadrilateral and their particular contributions to mathematical history.

A Saccheri Quadrilateral is a figure consisting of two right base angles and two congruent sides opposite each other and adjacent those base angles (Figure 1). A Saccheri Quadrilateral is also known as an isosceles quadrilateral or isosceles birectangle because of the similarities to an isosceles triangle. Each figure has congruent sides and base angles. The Saccheri Quadrilateral first arose from attempts by Omar Khayyamin and Nasir Eddin al-Tusi to create a rectangle in neutral geometry, but was made famous by a Jesuit priest named Girolamo Saccheri.<sup>1</sup>

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<sup>1</sup> Marvin Jay Greenberg, *Euclidean and Non-Euclidean Geometries*, Third Edition, W.H. Freeman and Co., New York, NY, 1993, pg. 155.

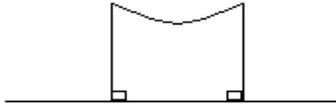


Figure 1: A Saccheri Quadrilateral

Saccheri proposed that there were only three cases for the summit angles of said quadrilateral:

Case 1: The summit angles are right angles.

Case 2: The summit angles are obtuse angles.

Case 3: The summit angles are acute angles.

Saccheri felt if he could show that the first case was the only possible, reasonable case that he would have proved the Euclidean Parallel Postulate which, paraphrased, says that there can only be one line parallel to another through a given point. It is fairly simple to show that the first case is possible. If one assumes Euclid V to be correct, then lines with a common perpendicular are parallel. (Figure 2) This would make the quadrilateral a rectangle, which was what Saccheri wanted to achieve.



Figure 2: A Saccheri Quadrilateral with right summit angles becomes a rectangle

Saccheri then showed the second case to be impossible. He found that if the summit angles were obtuse then this implied Euclid V, which implied right angles and

led then to a contradiction with the hypothesis.<sup>2</sup> The last case, however, stumped Saccheri, who in the end concluded that: “The hypothesis of the acute angle is absolutely false, because it is repugnant to the nature of the straight line.”<sup>3</sup> What Saccheri did not realize was that he was on the fringes of hyperbolic geometry and could therefore never prove his third case to be either true or false.

Saccheri actually concluded that his acute angle hypothesis showed that there exist two straight lines that merge into one straight line at infinity and have a common perpendicular at infinity (Figure 3).<sup>4</sup> However, he was too bound by tradition to consider a non-Euclidean geometry, and as a result did not realize he had virtually uncovered spherical geometry. While this figure seemed absurd to him, he was not convinced by his proof that Euclid V did not hold absolutely.

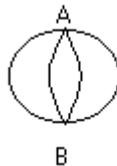


Figure 3: Saccheri’s conclusion of two straight lines that merge into one straight line at infinity.

Approximately 40 years after Saccheri, another attempt was made to create a rectangle in neutral geometry. Johann Lambert used the idea of an Egyptian Scientist ibn-al-Haytham to create a quadrilateral with three right angles to try and force the fourth angle to be right as well (Figure 4). Using this quadrilateral Lambert was able to

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<sup>2</sup> H.S.M. Coxeter, *Non-Euclidean Geometry*, Sixth Edition, Mathematical Association of America, Washington D.C., 1998, pg. 5.

<sup>3</sup> Greenberg, 1993, pg. 155.

<sup>4</sup> Harold E. Wolfe, *Introduction to Non-Euclidean Geometry*, Holt, Rinehart and Winston, Inc., New York, NY, 1945, pg. 32.

disprove an obtuse fourth angle, but, just as Saccheri, was foiled by the proof that acute angles could not occur. His observations of the acute angle hypothesis led him to the conclusion that on a sphere of imaginary radius the acute angle could be verified.<sup>5</sup> Sommerville defines a sphere as: “In space of three dimensions the locus of a point which is equidistant from a fixed point.”<sup>6</sup> When that fixed point is at infinity, the sphere becomes a horosphere. When the center is an ideal point, the sphere is called an Equidistant surface. While geometry on a sphere with a real vertex is elliptical, Lambert’s results helped show that geometry on an equidistant sphere is hyperbolic.<sup>7</sup> But, Lambert wanted to prove create the rectangle in two dimensions using Neutral geometry, and couldn’t. Although he didn’t realize it at the time, he was studying hyperbolic geometry and his discoveries, published posthumously, would fuel later research in the area.<sup>8</sup>

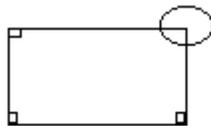


Figure 4: A Lambert Quadrilateral has three right angles and the fourth angle is undesignated.

While they didn’t realize it at the time, both Saccheri and Lambert were entering into the realm of hyperbolic geometry, a realm where parallelism was not the same as it had been for the two thousand years since Euclid presented his geometry. Both of the presented quadrilaterals take on new and interesting properties in hyperbolic geometry.

<sup>5</sup> Duncan M. Y. Sommerville, *The Elements of Non-Euclidean Geometry*, G. Bell and Sons, London, UK , 1914, pg. 13.

<sup>6</sup> Sommerville, 1914, pg. 52.

<sup>7</sup> Sommerville, 1914, pg. 56.

<sup>8</sup> Greenberg, 1993, pg. 161.

What distinguishes hyperbolic geometry from Euclidean geometry is a single parallel axiom. In Euclidean geometry, Euclid's fifth proposition holds. This proposition is equivalent to an axiom presented by David Hilbert in the late 1800's stating that there can be at most one line through a point not on a given line that is parallel to the original line. Hyperbolic Geometry is based on the negation of Hilbert's axiom; more than one line through a given point parallel to the original line can exist.<sup>9</sup> When one considers this change in the basis of geometry, many interesting and intriguing deductions make themselves apparent. The Saccheri and Lambert quadrilaterals are no exception to this rule.

As stated above, in hyperbolic geometry a rectangle can not exist.<sup>10</sup> Yet, Lambert and Saccheri did everything in their power to try and force one to exist. While he did not know he was doing so, Saccheri deduced many of the theorems that became foundations for non-Euclidean geometry.<sup>11</sup> With regards to his quadrilateral (refer back to Figure 1), Saccheri made some discoveries that went against many previously concluded theorems. If one were to construct an isosceles quadrilateral  $ABCD$  with right base angles  $\angle A$  and  $\angle B$ , and then construct acute summit angles, segment  $CD$  must then be longer than segment  $AB$ , and segments  $AB$  and  $CD$  will only have a common perpendicular at the midpoint (Figure 5). This idea of a single common perpendicular flies in the face of anyone used to Euclidean Geometry.

While interesting things can be derived from the Saccheri Quadrilateral, even more can be learned from the Lambert Quadrilateral. If there can not be a rectangle in

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<sup>9</sup> Greenberg, 1993, pg. 187.

<sup>10</sup> Greenberg, 1993, pg. 187.

<sup>11</sup> Wolfe, 1945, pg. 32.

hyperbolic geometry, then by the law of the excluded middle the fourth angle in a Lambert Quadrilateral (refer back to Figure 4) must be either acute or obtuse. Using the same logic that Lambert used to show that the angle could not be obtuse in Euclidean geometry one can show that the angle can not be obtuse in hyperbolic geometry.

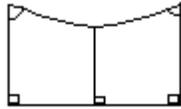


Figure 5: A Saccheri Quadrilateral with acute summit angles.

Therefore, the angle must be acute. But, when one looks at the angle of the diagonal to the non-right angle, one learns that the fourth vertex does not exist, but is an ideal point at infinity (Figure 6).<sup>12</sup> This is consistent with Lambert's discovery of the sphere of imaginary radius. The two sides adjacent to the undetermined angle act as asymptotic rays to the diagonal and only come into contact at infinity. It is from this conclusion that Lambert discovered the Equidistant surfaces. In Euclidean geometry a triangle formed by the arcs of a great circle has an angle sum greater than  $180^\circ$ . The excess above  $180^\circ$  is proportional to the area of the triangle, with the constant of proportionality being  $r^2$ , or the radius of the sphere. To find triangles with less than  $180^\circ$  Lambert speculated that if  $r$  were replaced by  $ir$  (where  $i$  is  $\sqrt{-1}$ ) then the proportionality constant would be negative. From this conclusion the sphere of imaginary radius was deduced.<sup>13</sup> The point at infinity had at that time not been considered in geometry, and that is why Lambert found the idea embarrassing and did not publish his discoveries.<sup>14</sup>

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<sup>12</sup> Greenberg, 1993, pg. 413.

<sup>13</sup> Greenberg, 1993, pg. 160.

<sup>14</sup> Sommerville, 1914, pg. 14.

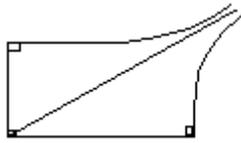


Figure 6: A Lambert Quadrilateral with the fourth vertex at infinity.

The historical progression of hyperbolic geometry was a 2000-year process. With the foundation set by Euclid, Geometers around the world worked to solve the mysteries of the Universe. Unfortunately for them and fortunately for us the foundation that Euclid set could not solve all the mysteries. Lambert and Saccheri were two men whose studies furthered the science of mathematics through both their successes and their “failures.” The quadrilaterals that they studied have become an integral part of the modern day study of geometry. As such, they should be revered in the annals of history as partial founders of non-Euclidean Geometry, and their respective quadrilaterals looked at as launching points for the study of such geometries.

### References Cited

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