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Geometry 300

Saccheri Quadrilaterals: The Discovery of Non-Euclidean Geometry

> *The value of non-Euclidean geometry lies in its ability to liberate us from preconceived ideas in preparation for the time when exploration of physical laws might demand some geometry other than the Euclidean.*
>
> *-Bernhard Reimann*

Throughout the centuries many mathematicians have tried to prove Euclid's parallel postulate. One such Logician was the Jesuit priest Girolamo Saccheri (1667-1733). Saccheri was a Professor of Grammar in Milan, lectured on Philosophy in Turin, and in Mathematics and Theology in Pavia and most of his books concerned theology. Before he died, he published a book entitle *Euclides ab omni naevo vindicatus (Euclid Freed of Every Flaw)*.[1] His book went unnoticed for over a century and half until the Italian mathematician Eugenio Beltrami discovered it. Saccheri wished to prove Euclid's fifth postulate using a *reductio ad absurdum* argument, that is, by using other axioms to prove the postulate. He assumed the negation of the Parallel Postulate and tried to arrive at a contradiction.[2] Euclid's fifth postulate runs:

> If two lines are intersected by a transversal in such a way that the sum of the degree measures of the two interior angles on one side of the transversal is less than 180°, then the two lines meet on that side of the transversal.[3]

In order to prove this, Saccheri studied a family of quadrilaterals that was studied many centuries earlier by the poet Omar Khayyam (1048-1131), who started with two parallel lines $A B$ and $D C$, formed the sides by drawing lines $A D$ and $B C$ perpendicular to $A B$,

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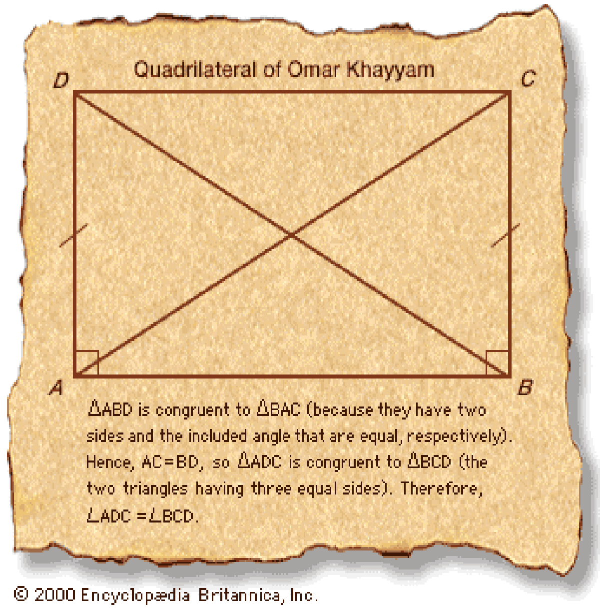
[1] Greenberg 154
[2] "Saccheri" www.math.uncc.edu/~droyster/math3181/notes/hypgeom/node41.html



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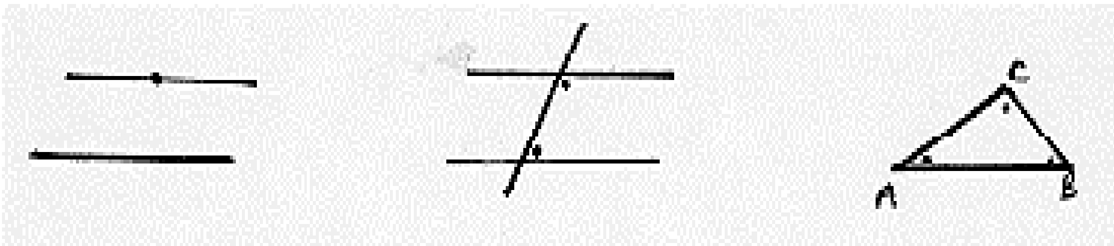
and then Khayyam considered three hypotheses for the internal angles at *C* and *D*: to be either right, obtuse, or acute (pictured to the right).[4] These quadrilaterals have come to be known as *Saccheri quadrilaterals*.

Euclid's postulate number V, known as the Parallel Postulate, was flawed by the fact that was incomplete. At present, in order for a postulate to be considered valid, it needs to follow three major requirements.

> CONSISTENCY so that it is impossible to derive two contradictory theorems from postulates,
> INDEPENDENCE so that no postulate can be derived from the others, and
> COMPLETENESS so that everything that will be used to derive the theory is stated in the premises, leaving no tacit assumptions.[5]

For twenty-one centuries, geometers tried to prove the postulate and they even expressed it in many different ways in an effort to make it less intimidating. The three cases that Khayyam and then later Saccheri considered for their hypothesis are shown in the following figure: (from left to right) "Through a point in a plane one unique line can be drawn parallel to a given line", "If a straight line falling on two straight lines makes the interior angles on the same side together less than two right angles, the two straight lines, if produced indefinitely, meet on that side which the angles are together less than two right angles" (this is Euclid's version), "The angles of a triangle sum to 180°." It has

[3] Greenberg, 128
[4] "Non-Euclidean Geometries" Encyclopedia Britannica, Inc., 2000
[5] MacDonnell, Joseph. "Girolamo Saccheri, S.J. (1667-1733) and his solution to Euclid's blemish" www.faculty.fairfield.edu/jmac/sj/sacflaw/sacflaw.htm, 4



been found that Euclid was correct in not providing a proof for postulate V because there is none: it is independent of the other four postulates as well as his first twenty-eight theorems.[6] Eugenio Beltrami discovered this in 1868 by exhibiting a model of non-Euclidean geometry that existed in Euclidean Geometry.

Saccheri was the first geometer to impose laws of logic in his attempt to eliminate the flaw in Euclid's proof.[7] He formulated the problem in terms of three hypotheses, only one of which can be correct.

Case 1: The summit angles are right angles.
Case 2: The summit angles are obtuse.
Case 3: The summit angles are acute.

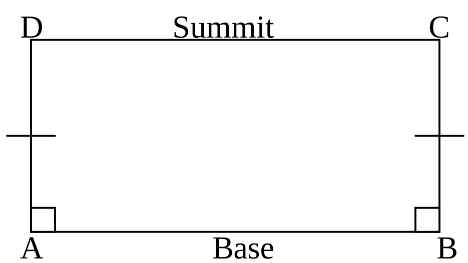


Figure: Saccheri Quadrilateral

Saccheri wanted to prove the first case, so in order to prove Case 1 Saccheri set out to contradict the other two cases. He succeeded in showing that Case 2 leads to a contradiction: if the summit angles were obtuse, the angle sum of the quadrilateral would be more than 360°, contradicting the corollary 2 to the Saccheri-Legendre theorem (See

[6] MacDonnell, 5
[7] MacDonnell, 6

Appendix A).[8] As Saccheri called it "the inimical acute angle hypothesis," he found it impossible to find a contradiction in order to solidify Euclid V. Saccheri penetrates far deeper into Lobacevskian geometry than his predecessors to contradict case 3. Nikolai Ivanavic Lobacevskii was the first to publish a paper, "On the Principles of Geometry," about the discovery of non-Euclidean geometry, which he called *imaginary geometry*. He introduced the basic concepts of geometry that do not depend on the parallel postulate.[9]

> [Saccheri] shows that under the acute-angle hypothesis two straight lines can intersect, or have a common perpendicular on each side of which they diverge, or diverge in one direction and come asymptotically close to one another in the other direction. In the latter case, Saccheri concludes that these straight lines must have a common point and a common perpendicular at infinity.[10]

Though he was able to produce many results, he was not pleased. He exclaimed in frustration, "The hypothesis of the acute angle is absolutely false, because [it is] repugnant to the nature of the straight line!" Saccheri declared his dissatisfaction with a concluding remark:

> It is well to consider here a notable difference between the foregoing refutations of the two hypotheses. For in regard to the hypothesis of obtuse angle the thing is clearer than midday light…
> But on the contrary I do not attain to proving the falsity of the other hypothesis, that of acute angle, without previously proving that the line, all of whose points are equidistant from an assumed straight line lying in the same plane with it is equal to this straight line.[11]

Although he had not realized it, Saccheri had discovered non-Euclidean geometry. Whether the right, obtuse, or acute hypotheses are true, the sum of the angles of a triangle

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[8] Greenberg, 155
[9] Rosenfeld, B. A. A History of Non-Euclidean Geometry: Evolution of the Concept of a Geometric Space. "Lobacevskian Geometry" Springer-Verlag Inc: New York, 1998. 206
[10] Rosenfeld, "Saccheri's theory of Parallel lines." 98.
[11] Rosenfeld, 99

respectively equals, exceeds, or falls short of 180°. Some of the theorems that followed from Saccheri quadrilaterals are listed below:

1. The summit angles of a Saccheri quadrilateral are equal and acute. (Proof in Appendix B)
2. The line joining the midpoint of the base and summit – called the altitude – of a Saccheri Quadrilateral is perpendicular. (Proof in Appendix B)
3. Two Saccheri quadrilaterals are congruent if they have equal summits and equal summit angles.
4. The sum of the angles of any triangle is less than two right angles.
5. If the angles of a triangle are equal to the angles of another then the two triangles are congruent. [12]
6. In a Saccheri quadrilateral the summit is greater than the base and the sides are greater than the altitude. [13]

Instead of leading to a proof for postulate V, Girolamo Saccheri uncovered the consequences of the acute angle hypothesis. Though Saccheri carefully plotted a course through his proofs, his 33rd theorem contains a flaw. He breaks away from his rigorous logic and remarks, "…but this is contrary to our intuitive knowledge of a straight line." Saccheri ends his book by admitting that he has not completely proven the "acute case" and for this reason is said to have withheld publication of the book during his lifetime.[14]

> I do not attain to proving the falsity of the other [acute angle] hypothesis without previously proving that the line, all of whose points are equidistant from an assumed straight line lying in the same plane with it, is equal to this straight line, which itself finally I do not appear to demonstrate from the viscera of the very hypothesis, as must be done for a perfect refutation…. But this is now enough.

Alberto Dou, S.J., who has studied Saccheri's Euclides for years, demonstrates that Riemann, Lobachevsky, Bolyai and Gauss not only had direct or indirect access to Saccheri's Euclides but they also used his method and theorems. Duo also notes that

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[12] s13a.math.aca.mmu.ac.uk/Geometry/M23Geom/NonEuclideanGeometry/NonEuclidean, 2 (theorems 1-5)
[13] www.math.uncc.edu/~droyster/math3181/notes/hyprgoem/node41.html, 3
[14] MacDonnell, Joseph. "Theorems of Girolamo Saccheri, S.J. (167-1733) and his hyperbolic geometry. www.faculty.fairfield.edu/jmac/sj/sacflaw/sacther.htm, 7

among Saccheri's flaws, he includes the assumptions that all lines are infinite in length, that in every case the exterior angle is greater than an interior angle and that a point at infinity possess the same properties as an ordinary point.[15]

Today, the non-Euclidean geometries are referred to as elliptic geometry and hyperbolic geometry. Elliptic geometry involves the new topological notion of "nonorientability," since all the points of the elliptic plane not on a given line, lie on the same side of that line.[16] The difference between hyperbolic and Euclidean geometry lies in Hilbert's parallel postulate, which is equivalent to Euclid's parallel postulate. Euclidean geometry includes all the axioms on neutral geometry and Hilbert's parallel postulate whereas hyperbolic geometry assumes all the axioms of neutral geometry and the negation of Hilbert's parallel postulate, which is sometimes called the "hyperbolic axiom." Saccheri quadrilaterals are used in many of the proofs for the theorems in hyperbolic geometry.

In spite of his abrupt conclusion, Saccheri's investigation of Euclid V and quadrilaterals was a crucial step towards the evolution of non-Euclidean geometries. He broke ground for future geometers, which can now be realized as his major achievement. But many past historians missed Saccheri's contribution because earlier writers viewed the scholarly works of the Jesuit order with contempt. Unfortunately, Saccheri failed himself in not being able to realize the true significance of his discovery and moving beyond the belief that Euclid's was the only true geometry.

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[15] MacDonell, Theorems of Giolamo Saccheri…, 7
[16] Greenberg 2

Appendix A

Saccheri-Legendre Theorem is as follows: The sum of the degree measures of the three angles in any triangle is *less than or equal to 180°*.[17]

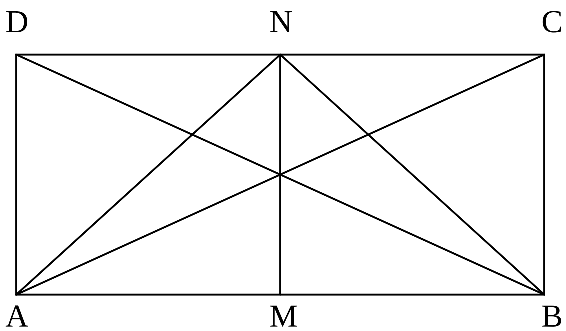
Corollary 2 to the Saccheri-Legendre theorem: The sum of the degree measures of the angles in any *convex* quadrilateral is at most 360°.[18]

Quadrilateral ABCD is called *convex* if it has a pair of opposite sides, e.g., AB and CD, such that CD is contained in one of the half-planes bounded by line AB and segment AB is contained in one of the half-planes bound by line CD.[19]

Appendix B

Theorem: In a Saccheri Quadrilateral

1) The summit angles are congruent, and
2) The line joining the midpoints of the base and the summit is perpendicular to both.



Proof: Let M be the midpoint of AB and let N be the midpoint of CD

1. We are given that (∠DAB)° = (∠ABC)° = 90°.
   Now, AD≅BC and AB≅AB, so that by SAS ΔDAB≅ΔCBA, which implies that BD≅AC. Also, since CD≅CD then we may apply the SSS criterion to see that ΔCDB≅ΔDCA. Thus, ∠D≅∠C.

2. We need to show that line MN is perpendicular to both line AB and line CD. Now DN≅CN, AD≅BC, and ∠D≅∠C, thus by SAS ΔADN≅ΔBCN. This means that AN≅BN. Also, AM≅BM and MN≅MN. By SSS ΔANM≅ΔBNM and it follows that ∠AMN≅∠BMN. They are supplementary angles; hence they must be right angles. Therefore line MN is perpendicular to line AB.

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[17] Greenberg, 125
[18] Greenberg, 127
[19] Greenberg, 127

Using the analogous proof and triangles ΔDMN and ΔCMN, we can show that line MN is perpendicular to line CD.[20]

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[20] www.marh.uncc.edu/~droyster/math3181/notes/hyprgeom/node.html, 1