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Hyperbolic Geometry and the Universe

Throughout our history, men have pondered the universe in which we live and the nature of that universe. In the twenty first century we are in a unique position to answer the questions of the cosmos. Advances in mathematics and physics have given insight into how the universe began, how big it is, how it is shaped, and how if ever it will end. Mathematicians and physicists have recently devoted increased attention to the questions surrounding the shape of the universe. Geometry and topology are our primary theoretical tools for formulating conjectures concerning the shape of the universe that can be tested experimentally in the near future with more advanced technology. It is widely regarded that our universe has one of three geometries however there are many different theoretical topologies consistent with mathematical theory and our limited range or experimentation. Hyperbolic geometry offers the widest range of topological options that could describe our universe and will be the concentration of the following discussion.

The sole difference between the axioms of Euclidean and hyperbolic plane geometry is the parallel axiom. Since antiquity, the Euclidean parallel postulate was widely thought to be true but could not be proven from the axioms of neutral geometry. Beginning in the 19^{th} century, mathematicians began exploring the possibility that the Euclidean parallel postulate was false. If "the Euclidean parallel postulate is false" is taken as an axiom of geometry we are now working in hyperbolic geometry. The axioms of hyperbolic geometry are just as logically consistent as those of Euclidean geometry, however the possible shapes of a universe that admits Euclidean geometry are very different than those of a universe that admits hyperbolic geometry.

Sometime in the 3rd century B.C. Euclid, a Greek mathematician working in Alexandria, authored a now famous mathematical treatise entitled *Elements*. Included in this work were his axioms for geometry and an extensive collection of deductions following logically from the axioms. (Osserman pgs.5-6) In *Elements*, Euclid made many implicit assumptions, some based on diagrams, which needed to be explicitly stated in order for geometry to be a true axiomatic system. In the early twentieth century, David Hilbert extensively reworked Euclid's geometry making it logically consistent. Hilbert's geometry is essentially the same as Euclid's and the important thing is that Hilbert's geometry takes Euclid's parallel postulate as an axiom because it cannot be proven from the other axioms.

Hilbert's Parallel Axiom for Euclidean Geometry: For every line l and every point P not lying on l there is at most one line m through P such that m is parallel to l.

Theorem 4.1 Corollary 2 (Greenberg pg. 117) of neutral geometry states that there is at least one line through P parallel to l. The combination of these two statements gives us exactly one line through P parallel to l. As stated earlier, if Hilbert's parallel axiom is replaced by the negation of that axiom we have an independent axiomatic system called hyperbolic geometry. (Greenberg pg.102)

Universal Hyperbolic Theorem: In hyperbolic geometry, for every line l and every point P not lying on l, there are at least two lines through P parallel to l.

Proof

1.	There is a line l and a point P not on l	1. Given

2.	Drop perpendicular from *P* to *l*, call the foot *Q*	2. Prop 3.16
3.	Construct line *m*, through *P* perpendicular to line *PQ*	3. Prop 3.16
4.	Line *m* is parallel to line *l*	4. Cor. 1 Thm. 4.1
5.	There is another point *R* on *l*	5. BA 2
6.	Construct line *t*, through *R* perpendicular to *l*	6. Prop 3.16
7.	Drop perpendicular from *P* to *l*, call the foot *S*	7. Prop 3.16
8.	Line *PS* is parallel to line *l*	8. Cor. 1 Thm. 4.1
9.	Suppose *S* is incident with *m*	9. RAA hypothesis
10.	Then quadrilateral *PQRS* is a rectangle	10. Defn. Rectangle
11.	We have a contradiction	11. Thm. 6.1
12.	*S* is not incident with *m*	12. RAA conclusion
13.	Line *PS* is not line *m*	13. Step 12, IA 1
14.	There are two lines through *P*, parallel to *l*	14. Step 4, 8, 13

Corollary: There are an infinite number of lines through *P* parallel to *l*.

Proof

We know from betweenness axiom one that there are an infinite number of points R_1, R_2, R_3, … on line *l*. We can repeat steps 5 through 14 of the above proof for each point R_i on line *l*. Because we assumed only that *R* is not *Q* and incident with *l*, as long as R_i satisfies these conditions it will yield another line parallel to line *l*. Therefore an infinite number of points implies an infinite number of lines parallel to line *l*. (These proofs are taken directly from Greenberg pg.187-8)

In Euclidean geometry, Hilbert's parallel axiom is taken as an axiom and in hyperbolic geometry the negation of Hilbert's parallel axiom is taken as an axiom. This is the only difference between the two separate geometries. We know by the meta-mathematical theorem in Greenberg that if Euclidean geometry is consistent so is hyperbolic geometry. (Greenberg pg. 225) This equivalent consistency leads one to wonder whether our universe, despite its Euclidean appearance on small scales, could be hyperbolic and what consequences hyperbolic geometry implies. A good way to begin analyzing these consequences is to study the behavior of the angles of simple polygons.

The Saccheri-Legendre theorem in neutral geometry states that the sum of the angles of a triangle must be less than or equal to π radians. (Greenberg pg. 125) From this theorem and its corollaries we can deduce that the sum of the angles of any convex polygon is less than or equal to π x (n – 2) where n = the number of sides of the polygon.

Proof by induction:

1. The smallest polygon, a triangle has angle sum π x (3 – 2) = π radians by Saccheri-Legendre Theorem
2. Suppose a convex polygon of an arbitrary number m sides has angle sum π x (m – 2) radians.
3. Any convex polygon of m + 1 sides can be divided into a convex polygon of m sides and a triangle by constructing a line segment between proper vertices.
4. π x (m – 2) + π radians = π x (m + 1 – 2) radians.

The first case holds and the m^{th} case implies the $(m + 1)^{th}$ case so we have an induction proof.

Because the Saccheri-Legendre Theorem holds in neutral geometry it holds in both Euclidean and hyperbolic geometries. (Greenberg pg. 117) There is another homogeneous geometry, elliptic geometry where triangles have angle sums greater than π radians. Elliptic geometry will not be discussed at length in this paper. If a triangle exists whose angle sum is π radians, then a rectangle exists. If a rectangle exists, then every triangle has angle sum equal to π radians. (Theorem 4.7 Greenberg pg. 131) By proposition 4.11 of Greenberg, we know that Hilbert's parallel postulate is logically equivalent to "the angle sum of every triangle is π radians." (Greenberg pg.130) From these theorems we see that all triangles have angle sum π radians in Euclidean geometry and all triangles have angle sum less than π radians in hyperbolic geometry.

The properties of these angle sums become important when we explore the topologies that admit Euclidean and hyperbolic geometries. The topology of a surface is the properties of the surface that are unaffected by stretching, twisting, or otherwise deforming that surface. However tearing, poking holes in, and other violent actions change the topology of a surface. (Weeks pg. 28) Some examples of different topologies are a sphere, a torus, and tori with any number of holes. A homogenous surface is one whose local geometry is the same at all points. (Weeks pg.44) Topologically speaking any surface can be given a homogenous geometry. (Weeks pg.44)

Picture abstractly gluing the sides of a two dimensional polygon to form a surface. The angles of the polygon must have sum equal to 2 π radians in order for the surface to be homogeneous. If the angles are too small we will have a cone and if they are too large they simply will not fit together smoothly. Surfaces formed by abstractly gluing the sides of a quadrilateral admit Euclidean geometry because the angles of a quadrilateral have sum 2 π radians in Euclidean geometry and smoothly fit together. Consider a larger sided polygon, a pentagon for example. Pentagons have angle sum 3 π radians in Euclidean geometry and less than 3 π radians in hyperbolic geometry. As the pentagon's area grows in the hyperbolic plane each angle gets smaller, as its area approaches infinity each angle approaches zero. If we stretch the pentagon, which does not change its topology, until its angle sum is 2 π radians, we can abstractly glue the corners together to form a perfectly homogenous surface. (Weeks ch. 11) The topology of a sphere admits elliptic geometry, a torus admits Euclidean geometry, and almost any other topology admits hyperbolic geometry. This is because with the exception of a sphere and a torus, surfaces are topologically equivalent to the result of gluing edges on a many (more than four) sided polygon. We can see that unless our universe has the topology of a sphere or a torus it will admit hyperbolic geometry. The topological possibilities are infinite in hyperbolic geometry.

A surface like a sphere or a torus is called a two dimensional manifold (two-manifold for short) because although the surface is embedded in three space it intrinsically has only two dimensions. If someone walked around on the inside or outside of a sphere or torus or any other two-manifold she could only move two dimensionally (left, right, forward, or back) without damaging or leaving the surface. Our universe is a three-manifold because it intrinsically contains three dimensions. We need to think in terms of at least three-manifolds to adequately describe our universe. A three-manifold may or may not be embedded in four-space the way we know two-manifolds to be embedded in three space, but this issue is not of central concern. Consider a circle for simplicity, which is a one-manifold because it has only one intrinsic dimension. To move to the two-manifold we add another dimension but retain the circular geometry. We have a circle of circles, which is known as a torus. To move to the three-manifold we follow the same procedure and end up with a circle of tori. It may not be clear what a circle of tori is but it can be explained in the following way. The outside of the torus is abstractly glued to itself; that is if one left the torus at a point she would return at a corresponding point on the other side. The point of exit and the point of entry are indeed the same point. Three-manifolds have the same geometry and topology as their corresponding two-manifold whose properties are easily understood. (Weeks ch.14)

The reason we are primarily concerned with homogeneous manifolds is that they are most likely to describe our universe. The experiments of physicists lead us to believe that we live in an isotropic universe; that is "no matter where you are in the universe things look basically the same in all directions."(Weeks pg. 267) "If we look at distant galaxies [in every direction], there seems to be more or less the same number of them. So the universe does seem to be roughly the same in every direction, provided one views it on a scale large compared to the distance between galaxies and ignores differences on a small scale." (Hawking pg.43) If our universe were non-homogeneous it would exhibit different geometries in different places. If one was standing somewhere between a region that was Euclidean and a region that was hyperbolic things would not be basically the same in all directions. In fact they would be very different. Therefore it is a necessary condition that an isotropic manifold is also homogeneous. (Weeks pg. 267) As seen earlier almost all topologies admit hyperbolic geometry so the vast majority of homogenous surfaces admit hyperbolic geometry.

In 1929, Edwin Hubble used the world's largest telescope to experimentally confirm the prediction from Einstein's general relativity that the universe is expanding. Every galaxy in the universe is moving away from every other galaxy at a rate proportional to the distance between them. (Osserman pg. 104-5) It is questionable whether the universe will continue expanding forever or whether it will slow down and eventually collapse on itself in a 'big crunch.' "To answer this question we need to know the present rate of expansion of the universe and its percent average density. If the density is less than a certain critical value, determined by the rate of expansion, the gravitational attraction will be too weak to halt the expansion. If the density is greater than the critical value, gravity will stop the expansion at some time in the future and cause the universe to recollapse." (Hawking pgs.48-49)

General relativity predicts different answers to the question based on the geometry of the manifold that represents the shape of the universe. Universes with elliptic geometries will eventually stop and recollapse. A Euclidean universe would expand slower and slower but never stop. A hyperbolic universe will expand at a greater and greater rate. (Weeks pg. 273-4) If physicists can accurately measure the rate of expansion of the universe and approximate its size and mass to within a certain error, we will know what type of geometry our universe has. If physicists can, through other means such as measuring the angles of cosmic triangles, determine what type of geometry our universe exhibits we will know its fate. Current evidence suggests that the universe will expand forever. (Hawking pgs. 49-50)

It is amazing to think that such timeless question as the shape and fate of our universe can be answered as soon as our technology is sufficiently advanced to make accurate measurements of distant objects. Geometry and topology are excellent mathematical tools for working on the cosmological questions. By exploring the nature of all the possible manifolds consistent with Euclidean and hyperbolic geometry we can determine what properties imply what topologies. These properties can be compared with experimental results to determine the shape of our universe. As seen in the previous paragraph the shape and fate of the universe are inherently linked. These two properties illustrate the importance of geometry in answering the mysteries of the cosmos.

Bibliography

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