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Math 301

**Provability?**

Ideas that suggest conformity with the facts or with reality either as an idealized abstraction or in actual application to statements, ideas, acts etc. are truth (Webster's 987). Mathematicians approach this generalization in every aspect of logical understanding. Within the mathematical world though, truth isn't as important to an axiomatic system as much as provability is. Provability assesses a collection of truths and falsities to determine an eventual conclusion (usually a theorem). Logisticians like Hilbert, Russell, Legendre, and Euclid attained their fame as mathematicians by using sets of truths to derive and hence prove theorems in mathematical fields such as Geometry, Calculus, and Set Theory. Although proof has been denoted as the fundamental verification of a consistent system in mathematics, Kurt Godel, a modern logician of mathematics showed that there exist unprovable statements that are seen to be true within the formal system of arithmetic.

Some philosophers of mathematics argue that proof of systems in mathematics can only be done in a systematic and axiomatic method. In particular, each justified statement is thereby assessed its truth-value from the argument following, until the conclusion desired is reached. This axiomatic method of proof leaves no ambiguity in judgment of provability, and fully attributes a specific truth-value to a particular statement or theorem. Furthermore, the principles of inference as mathematical induction are invalid as a tool of proof. There has been evidence through Godel's discovery that statements in a complete formal system that are strictly conceived (unseen) are unprovable, even though they are true to the beholder.

In order to be able to show that mathematical reasoning has limits you've got to say very precisely what the axioms and methods of reasoning are that you have in mind. In other words, you have to specify how mathematics is done with mathematical precision so that reasoning becomes a clear-cut question. Hilbert put it this way: *The rules should be so clear, that if somebody gives you what they claim is a proof, there is a mechanical procedure that will check whether the proof is correct or not, whether it obeys the rules or not*. This system of proof checking is the heart of formalizing axiomatic systems (The Berry Paradox).

Godel's Incompleteness Theorem provoked many interesting questions: What are the limits of rational thought? Can we ever fully understand the machines we build? Or the inner workings of our own mind? How should mathematicians proceed in the absence of logical certainty about their results? Godel's theorem stated: *for every consistent formalization of arithmetic, there exist arithmetic truths that are not provable within that formal system* (Casti, DePuali 48). Godel was interested in the relationships between numbers, he showed how it would be possible to represent arithmetic statements within arithmetic itself. In short, he sketched his proposed theorem by mirroring all statements about relationships between natural numbers by using these same numbers themselves.

Let's consider an invention called the Chocolate Cake Machine. Using it, people are able to satisfy their cake eating fancies in the comfort of their own home (Casti, DePuali). This device is operated by shoveling the ingredients needed for a cake along with a recipe for the cake into one end. The machine would then process the ingredients in accordance with the instructions given by the recipe, eventually serving up the desired cake at the machine's output slot. A Chocolate Cake Machine should produce every conceivable chocolate cake depending on its reliability and totality of production. In this context, the totality relies on the descriptions of every conceivable chocolate cake in the world (Casti, DePuali).

Although it may seem far removed form any deep philosophical considerations about life, the universe, or anything else, the question of existence of a reliable and total Chocolate Cake Machine encapsulates one of the principle philosophical questions: Is it possible to prove every truth?

The cake test allows us to prove the truth-value of each cake sent through the machine. A statement (cake) is provable if and only if there is a recipe that can be followed by the Chocolate Cake Machine for actually making that cake (Casti, DePuali). Note that it is sufficient to provide the recipe and show that if you did feed it into the machine, the result would be something that satisfies the chocolate cake test.

For the universe of cakes, let

*Truths* = all cakes satisfying the chocolate cake test

*Proofs* = all recipes for actually making chocolate cakes with the Chocolate Cake Machine

Note that it is not necessary to determine the provability of a chocolate cake with the ingredients following the recipe. Rather, it is sufficient enough to provide the recipe and show that if you did feed the ingredients into the machine, the result would be something that satisfies the cake test(Casti, DePuali).

We should ask whether there are honest chocolate cakes in the Platonic universe of cakes for which no recipe can ever be given? Or can following a set of instructions actually produce every object that satisfies the chocolate cake test? In sum, is there any theoretical barrier that restricts the production of such a reliable and total Chocolate Cake Machine?

Godel's Theorem addresses these questions in the realm of arithmetic numbers. Ultimately, his punch line was that there is an unbridgeable gap between what is true (or what appears to be true) within a given framework and what we cam actually prove by logical means using that same system.

Kurt Godel conclusively showed that what's true and what are provable are two very different concepts of logic. Therefore, despite the finest efforts of cake makers around the world, the cake bibles are doomed forever to incompleteness. Specifically, there will always exist chocolate cakes that can be seen to be legitimate chocolate cakes, yet whose recipes can never be written down (Casti, DePuali). Similar to mathematical induction, a cake that is not represented (only projected) doesn't have a recipe to prove its truth within the formal system.

An interesting way to further analyze the truth, provability, and formalization of an axiomatic system is to adventure into the world of paradoxes. Godel analyzed the paradox of the liar in terms of provability. The paradox of the liar is:

"**This statement is false**."

Why is this a paradox? Whether you assume that the statement is true or false, the outcome will be opposite of your assumption. In particular, if the statement is true, it directly tells you the falsity. On the other hand, if the statement is false, it is actually true because of its explicit definition. In sum, the liar lives in a paradoxical world achieving no verification of true or false conception.

Godel modified the statement to “**This statement is not provable**.” Consequently, he brought a theorem to the math world that in sum shattered the hopes that logic would allow us to have a complete understanding of the universe, as well as axiomatic provability.

Godel’s interest came in the relationships between numbers. He showed that it would be possible to represent numbers of arithmetic within arithmetic itself. The symbol ‘2’ is represented by the arithmetic value of *two* demonstrates a simplified version of his coding. He proved his theorem using mirrored statements about the relationships of natural numbers by using the same numbers themselves (Casti, DePuali).

To understand Godel’s mirroring operation with numbers more clearly, he coded elementary logical signs in terms of natural numbers (see Figure 1) in order to form logical sentences in numerical sequence (Casti, DePuali). This type of coding is complicated, but very similar to the computer science-coding scheme known as ASCII. By Godel’s coding procedure, every possible proposition about the natural numbers can itself be expressed as a number, and this opened up the possibility of using arithmetic to examine its own truths (Casti, DePuali). The formation of mathematical statements was transformed into sequences of numbers.

Figure 1:

With reference to the paradox of the liar, Godel wanted to address such a statement in the framework of arithmetic. His numbering scheme allowed him to code the assertion of the liar in terms of arithmetic language. To see how this numbering scheme works, take the numbers 5 and 9. It is a fact that 5 proceed 9 in sequencing order. We know this by formal knowledge that 5 and 9 are symbols of the natural numbers, when translated into a formal system of sequence, 5 comes before 9. In this way a truth of the real world is translated, or mirrored, by a truth from number theory formal systems. Godel used a modification of this coding to represent all possible statements of arithmetic using the language of arithmetic itself. By coding the paradox in arithmetic, Godel rightfully demonstrated that provability doesn't come directly. The paradox is still a paradox in the arithmetic language of his code (Casti, DePuali). In general, he employed both the arithmetic as a mathematical object, and the arithmetic as an interpreted formal system to talk about itself.

Briefly, the main steps in Godel's proof involve six fundamental steps. As just mentioned, Godel formalized a numbering scheme in order to formulate a translation of proof and theorem in arithmetic (Godel Language). Thereby implementing a mirror statement about the natural numbers in arithmetic. He then took an interpretive approach to the paradox of the liar, whereby translating the statement in terms of provability rather than truth (Casti, DePuali).

Using the combination of the numbering scheme and mirroring of the paradox, Godel formed statements that could thus be examined with regard to provability using his interpreted numbering in the formalized system of arithmetic. By showing that “This statement is not provable” has a counterpart in arithmetic, its translation in Godel Language represented every possible formalization of arithmetic. With this construction of sentences, Godel arrived that each formed statement (in Godel language) must be true if the formal system has no truths without verification.

Graphically, consider the formal system of arithmetic (**M**). The entire square represents all possible statements that can be made about the natural numbers (see Figure 2). The square is initially gray, when a statement using the rules of the formal system is proven true, we color that portion white; a false statement is colored black (Casti, DePuali). Godel’s theorem says that there will always exist statements (**G**) that are doomed to the area of gray. This result holds for every possible formal system **M**, provided that **M** is consistent. Conclusively, given a formal system **M** that is consistent, there is at least one statement **G** that cannot be proved or disproved in **M**.

Figure 2:

Godel's proof included a no escape clause. He showed that when additional axioms are added to a system where a Godel Language sentence is provable, the new system will attain a sentence that cannot be proven. His construction of the statement in Godel Language that "arithmetic is not consistent" inevitably demonstrated that the statement is unprovable (Casti, DePuali). His ending conclusion was that arithmetic is too weak to show its consistency as a formal system (Casti, DePuali).

He found that justification of his conclusion reached was that *Arithmetic is not completely formalizable*. Overall, Godel was able to show that for any consistent formal system that allows us to express all statements of ordinary arithmetic, a statement like the paradox of the liar must therefore exist. Consequently, the formalization must be incomplete (Casti, DePuali).

Godel's Incompleteness Theorem shows that provability is a weaker notion than truth, no matter what axiom system is involved ...the bottom line turns out to be that in every consistent formal system powerful enough to express all relationships among whole numbers, then the existence of a statement that cannot be proved using the guidelines of the system is (<u>The Berry Paradox</u>). In other words, given *any* set of arithmetic axioms from a consistent system, some true mathematical statements exist that cannot be derived from the set.

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