Megan Baxter
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MATH 300

The Baxter Betweenness Interpretation

For the purpose of this paper we have attempted to create an interpretation of the undefined terms in which all the Betweenness axioms hold. The only constraint of the interpretation is that points must be interpreted as rays. Interest in this project was sparked after completing a homework problem for an undergraduate geometry class. The problem required stating the analogues of Betweenness axiom 2 and Betweenness axiom 3 and stating whether or not these analogues were true. In order to create the analogue of each axiom, points were interpreted as rays. In this interpretation Betweenness axiom 2 states that given any two distinct, coterminal rays **XB** and **XD** (emanating from point X), there exist coterminal rays **XA**, **XC**, and **XE** such that **XA**∗**XB**∗**XD, XB**∗**XC**∗**XD,** and **XB**∗**XD**∗**XE.** Both the analogue of Betweenness axiom 2 and the analogue of Betweenness axiom 3 were found to be false because they did not take into consideration opposite rays.

After completing this assignment, questions arose concerning whether or not it would be possible to create an interpretation, based on interpreting points as rays, where the analogues of all the Betweenness axioms were true. What follows are the steps that were taken in the attempt to find such an interpretation.

When the homework problem was originally investigated the following information was given:

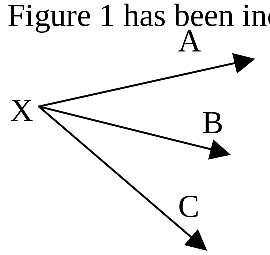
Definition: If A, B, and C are rays, then they are *coterminal* if they emanate from the same point. (Greenberg, 106)

Definition: Given an angle ∢AXC, define a point B to be in the *interior* of ∢AXC if B is on the same side of **XA** as C and if B is also on the same side of **XC** as A. (Greenberg, 81)

Definition: Let the notation A∗B∗C mean that ray B is between ray A and ray C. Ray **XB** is *between* rays **XA** and **XC** if **XA** and **XC** are not opposite rays and point B is interior to ∢AXC. (Greenberg, 82)

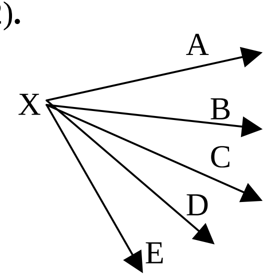
As mentioned previously, in order to create the analogues of each axiom, points were interpreted as rays. Using the criteria from above, the analogue of Betweenness axiom 1 states that if XA∗XB∗XC, then XA, XB, XC are distinct and coterminal and XC∗XB∗XA. This statement is pretty straightforward, and instead of going through a thorough proof, Figure 1 has been included to help illustrate this statement.

Figure 1:



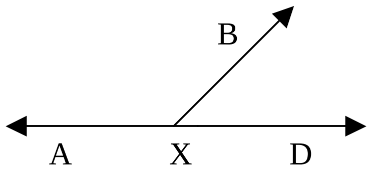
Let's now look at the analogue of Betweenness axiom 2. The analogue states that given any two distinct, coterminal rays **XB** and **XD** (emanating from point X), there exist coterminal rays **XA**, **XC**, and **XE** such that **XA**∗**XB**∗**XD, XB**∗**XC**∗**XD,** and **XB**∗**XD**∗**XE** (See Figure 2)**.**

Figure 2.



- **XA**∗**XB**∗**XD, XB**∗**XC**∗**XD,** and **XB**∗**XD**∗**XE** are all found to be false if ray **XA, XC,** or **XE** is either the opposite ray of **XB** or the opposite ray of **XD.**
    - If ray **XA** is the opposite ray of ray **XD**, then **XA**∗**XB**∗**XD** would not be correct. This situation fails the first part of the definition of betweenness for rays. Ray **XB** is between ray **XA** and ray **XD** if ray **XA** and ray **XD** are not opposite rays (See Figure 3).

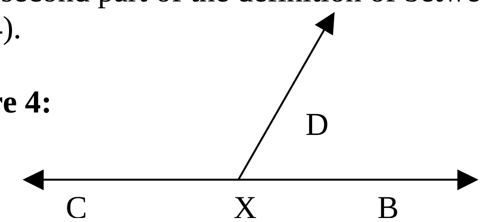
Figure 3:



- Similarly, if ray **XE** is the opposite ray of ray **XB** then **XB**∗**XD**∗**XE** is false.

- If ray **XC is** the opposite ray of either ray **XD** or ray **XB**, then **XB**∗**XC**∗**XD** are not correct. Point **C** is not interior to ∢**DXB,** and therefore the second part of the definition of betweenness for rays fails (See Figure 4).

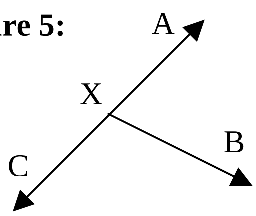
Figure 4:



Let us now look at the analogue of Betweenness axiom 3. As stated earlier, this analogue also fails because of opposite rays.

The analogue of Betweenness axiom 3 states that if ray **XA, XB,** and, **XC** are three distinct, coterminal rays, then one and only one of the rays is between the other two. This statement is false. The analogue does not take into consideration opposite rays. Consider Figure 5, where ray **A** and ray **C** are opposite rays.

Figure 5:



All three rays are distinct, and coterminal, however, no ray is between the other two. Ray **XB** is not between rays **XA** and **XC** because rays **XA** and **XC** are opposite rays and therefore fail the first part of the definition of betweenness for a ray. Ray **XA** is not between rays **XB** and **XC** because point **A** is not interior to ∢**BXC** and therefore does not satisfy the definition of betweenness for rays. Similarly **XC** is not between rays **XA** and **XB** because point **C** is not interior to ∢**AXB.**

Finally, let's look at the analogue of Betweenness axiom 4. This analogue states that for every line *l* and for any three rays **A**, **B**, and **C** not lying on *l*:

(1) If **A** and **B** are on the same side of *l* and **B** and **C** are on the same side of *l*, then **A** and **C** are on the same side of *l*.

(2) If **A** and **B** are on opposite sides of *l* and **B** and **C** are on opposite side of *l*, then **A** and **C** are on opposite side of *l*.

At this point we find ourselves in somewhat of a predicament. Before this point we have not had to consider lines and therefore do not have a definition for them. We also have not yet considered incidence or what it means for a ray to be or not to be lying on line *l*. To address these issues let us form an interpretation by defining the following terms.

B-Point: Ray emanating from O

B-Line: A particular side of line *l*

Let's stop here and clarify the definition of a B-line.

For the definition of a B-line consider a line *l* (*l* being used here as we commonly think of a line, as a dash) through point O (O being used as we commonly think of a point, as a dot). Line *l* has two sides. We will define a B-line as a particular side of line *l* but not both.

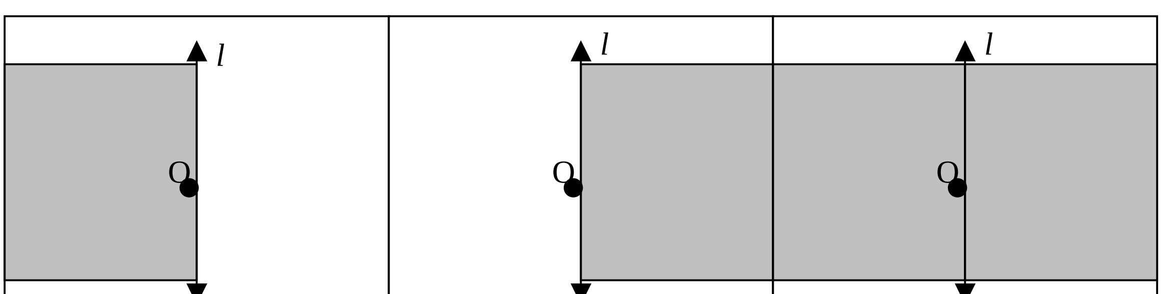
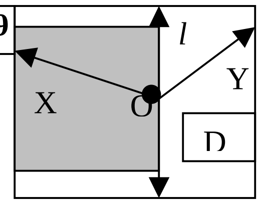


Figure 6 **Figure 7** **Figure 8**

Based on our definition of a B-line, in figures 6 and 7 the shaded area represents a B-line. In figure 8 the shaded area is not a B-line because it encompasses both sides of *l*. Now that B-lines and B-points are defined, B-incidence will be defined as follows:

Figure. 9



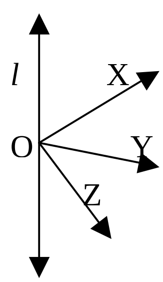
B-Incidence: Rays* on a particular side of line *l*.
(Remember that B-points are defined as rays.)

In Figure 9: B-point **OX** is B-incident with B-line, but B-point **OY** is not.

Now that B-line, B-point, and B-incidence have been defined, lets reevaluate all the Betweenness axioms and see if they are now true. Based on the new interpretation Betweenness axiom 1 states that if X∗Y∗Z, then X, Y, and Z are three distinct B-points all lying on the same B-line, and Z∗Y∗X.

Because we are given X∗Y∗Z we know from the definition of betweenness for rays that B-points X and Z are not opposite rays and the "point Y" (as in the dot) on B-point (ray OY) is interior to ∢**XOZ**. Based on these facts and the definition of a B-line all three rays are on a particular side of line *l* and are all incident with the same B-line. Z∗Y∗X can therefore be easily deduced from the previous information (See Figure 10).

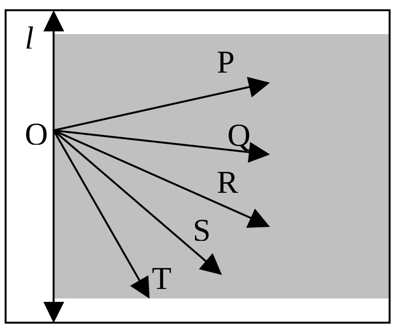
Figure 10:



Now that we know the analogue of Betweenness axiom 1 is still true in the new interpretation, lets see if the analogue of Betweenness axiom 2 is now true. Using the B-interpretation the analogue of Betweenness axiom 2 states that given any two distinct B-points Q and S on B-line *r*, there exists B-points P, R, and T lying on B-line *r* such that

P∗Q∗S, Q∗R∗S, Q∗S∗T. The problem that we had with the earlier analogue of Betweenness axiom 2 is that the interpretation did not take into consideration opposite rays. If ray **OP, OR,** or **OT** is either the opposite ray of **OQ** or the opposite ray of ray **OS** then P∗Q∗S, Q∗R∗S, and Q∗S∗T are all false. With the B-interpretation, however, a B-line is defined as a particular side of line *l*. All of the B-points in Betweenness axiom 2 are on B-line *r*. Therefore all the B-points are on one particular side of line *l* (See Figure 11). This means that opposite rays cannot exist because they would not be on the particular side of line *l* that all the other B-points are on. Therefore using the new interpretation the analogue of Betweenness axiom 2 is true.

Figure 11:
(The shaded area is B-line *r*)



Our problem with the previous analogue of Betweenness axiom 3 was also the fact that it did not take into consideration opposite rays. The B-interpretation of Betweenness axiom 3 states that if X, Y, and Z are three distinct B-points lying on the same B-line, then one and only one of the points is between the other two. It was shown earlier that if two of the B-points are opposite rays then no B-point is between the other two. With the B-interpretation, however, we do not need to worry about opposite rays. The hypothesis of Betweenness axiom 3 says that X, Y, and Z are distinct B-points **lying on the same B-line**. From our definition of B-line and B-incidence we know that B-points X, Y, and Z are all on one particular side of line *l*, so again opposite rays are not possible. This means that Betweenness axiom 3 is also true.

Recall, earlier in the paper we ran into some trouble when we tried to look at Betweenness axiom 4. Now that we have a new interpretation let us reevaluate this axiom and see if it is true. The analogue, based on the B-interpretation, of Betweenness axiom 4 states that for every B-line *r* and for any three B-points X, Y, and Z not B-incident with *r*:

(1) If X and Y are on the same side of *r* and Y and Z are on the same side of *r*, then X and Z are on the same side of *r*.

(2) If X and Y are on opposite sides of *r* and Y and Z are on opposite side of *r*, then X and Z are on opposite side of *r*.

Based on the definition of a B-line determined earlier it is impossible to distinguish between same-sidedness and opposite-sidedness. A B-line is defined as a particular side of line *l*. This means that a B-line is the entire area on that particular side of line *l*. Therefore there are not two distinct sides of a B-line like there is with an undefined line *l*.

The definition of a B-line was chosen because of the existing interpretation of points. B-points are defined as rays. When we simply just had points interpreted as rays Betweenness axiom 2 and Betweenness axiom 3 were false because they did not take into consideration opposite rays. By adding to the interpretation the definition of B-lines and B-incidence we were able to remove the threat of opposite rays. Although this allowed Betweenness axioms 2 and 3 to be true it made it impossible to show that Betweenness axiom 4 is true.

The goal of this paper was to take the original interpretation from the homework problem, points interpreted as rays, and modify it so that all Betweenness axioms are true. However, we must come to the conclusion that we are not able to do this. This does not mean that an interpretation, where all Betweenness axioms holds, is impossible to find. It simply means that an interpretation based on defining points as rays cannot satisfy all four of the Betweenness axioms.

References

Greenberg, M. J. Euclidean and Non-Euclidean Geometries; Development and History: 3rd Ed. W.H. Freeman and Company. New York. 2001