§ Chapter 1: Representations

1.1 Introduction

One of the many very useful mathematical concepts used in physics is a group. Groups have been so thoroughly studied that there is an extraordinary amount that can be said about a particular group's structure. Groups are wonderful tools for describing symmetries and thus a physicist might be inclined to exploit group structure to extract information about his or her problem. However, the mathematical definition of a group is just a set of elements that obey a few properties and these elements say nothing working in a real world coordinate system. A physicist may thus wish to find a representation for these group elements that is related to the coordinate system.

1.2 Representations

Consider $GL(V_n,C)$, the group of non-singular linear transformations of a vector space V_n. An element $T \in GL(V_n,C)$ is a map $T:V_n \to V_n$ over the complex numbers. Now if we define an ordered basis $\beta = \{\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_n\}$ for V_n, then every element $T \in GL(V_n,C)$ has an associated invertible square matrix with coefficients $a_{ij} = [T]_\beta$. There can be some confusion at this point because some authors define the **G**eneral **L**inear group $GL(V_n,C)$ in terms linear transformations, whereas others authors define the group in terms of their associated matrices. In this reading we use the former definition.

Definition 1.1 A *representation of* G is defined as a homomorphism $d:G \to GL(V_n,C)$. If d is injective (every element of the group has a unique matrix representation), then d is called a *faithful representation* of G. If we let $\beta = \{\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_n\}$ be an ordered basis for V_n, we then let D be the matrix of d with respect to this basis, $D = [d]_\beta$. We thus denote the coefficients of D as D_{ij}.

Recall that the homomorphic property just says that $d(g_1g_2) = d(g_1)d(g_2)\ \forall g_i \in G$, a very necessary property if we hope to accomplish anything. It's worth going over a few of the highlights of the above definition for understanding. When we defined a representation d we did this without specifying a basis for the vector space V_n. We could consider two different basis of V_n, say $\beta = \{\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_n\}$ and $\gamma = \{\vec{u}_1,\vec{u}_2,\ldots,\vec{u}_n\}$. The representation d can thus have two very different look associated matrices depending on the chosen basis, to whit $D = [d]_\beta$ and $D' = [d]_\gamma$. The two representations D and D' are called *equivalent* if they are related by a similarly transform S such that $D' = S^{-1}DS$, such as in this case.

Example Let's consider now an example of two representations of the group C_4. This group can be thought of as the group of 90-degree rotations about the z-axis. We denote its elements by $C_4 = \left(e, \rho_{\frac{\pi}{2}}, \rho_{\pi}, \rho_{\frac{3\pi}{2}}\right)$. Note also that this group is generated by the element $\rho_{\frac{\pi}{2}}$ and we therefore often denote a group in terms of its generator, $C_4 = \left\langle \rho_{\frac{\pi}{2}} \right\rangle$.

C_4	e	$\rho_{\frac{\pi}{2}}$	ρ_{π}	$\rho_{\frac{3\pi}{2}}$
Unfaithful rep.	1	1	1	1
Faithful rep.	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

The first representation, the unfaithful representation, is aptly called the *trivial* representation – notice that it does obey the homomorphic property and is a mapping to a one-dimensional vector space. The second representation is faithful because all of the elements are unique – notice that it too obeys the homomorphic property.

1.3 Invariant Subspaces

From linear algebra we need to recall the idea of a direct sum. Given a vector space V_n and two subspaces $W_k^{(1)}$ and $W_{n-k}^{(2)}$, then $V_n = W_k^{(1)} \oplus W_{n-k}^{(2)}$ if for every $\vec{x} \in V_n$ $\vec{x}$ can be written uniquely as $\vec{x} = \vec{w} + \vec{w}'$ where $\vec{w} \in W_k^{(1)}$ and $\vec{w}' \in W_{n-k}^{(2)}$. This means would mean that $W_k^{(1)} \cap W_{n-k}^{(2)} = \varnothing$ and $\dim\left(W_k^{(1)}\right) + \dim\left(W_{n-k}^{(2)}\right) = \dim(V_n)$. In terms of basis vectors, this is equivalent to saying that if $W_k^{(1)}$ is spanned by $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ and $W_{n-k}^{(2)}$ by $\{\vec{w}'_{k+1}, \vec{w}'_{k+2}, \ldots, \vec{w}_{n-k}\}$ then V_n is spanned by $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k, \vec{w}'_{k+1}, \vec{w}'_{k+2}, \ldots, \vec{w}_{n-k}\}$. When $V = W \oplus W'$, we call W' the *complement* of W in V. If $\vec{w}_i \cdot \vec{w}'_j = 0$ for all $\vec{w}_i \in W$ and $\vec{w}'_j \in W'$, we call W' the *orthogonal complement* of W in V.

Definition Given a representation $d: G \to GL(V_n, C)$, then if $\vec{v} \in V_n$ and $d(g)\vec{v} \in V_n$ for all $g \in G$, then we say V_n is *invariant* under G. Similarly, if $U_m \subset V_n$ is a proper subspace, $\vec{u} \in U_m$ and $d(g)\vec{u} \in U_m$ for all $g \in G$, then U_m is an *invariant* subspace under G.

Theorem Let $d: G \to GL(V_n, C)$ be a representation and $U_m \subset V_n$ a proper subspace. If U_m is an invariant subspace under G, then the orthogonal complement of U_m is also an invariant subspace under G.

Proof (sketch of ideas) For proof of this theorem we consider the representation $d: G \to GL(V_n, C)$ in matrix form D with respect to some basis γ. First we let $U_m^{(1)}$ be a G-invariant subspace of V_n and $\beta = \{v_1, \ldots, v_m\}$ be an ordered basis of $U_m^{(1)}$. We extend B to the ordered basis $\gamma = \{v_1, \ldots, v_m, \ldots, v_n\}$ of V_n and thus denote the subspace $U_m^{(2)}$ as being spanned by $\delta = \{\vec{v}_{m+1}, \ldots, v_n\}$. In this case we are not, in particular, looking at the *orthogonal* complement of $U_m^{(1)}$, but just the complement $U_m^{(2)}$. A representation takes on the form

$$D(g)=\left[\begin{array}{ccc|ccc} D_{11} & \cdots & D_{1m} & D_{1,m+1} & \cdots & D_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ D_{m1} & \cdots & D_{mm} & D_{m,m+1} & \cdots & D_{mn} \\ \hline D_{m+1} & \cdots & & & & \\ \vdots & & & & \ddots & \\ D_{n1} & \cdots & D_{nm} & D_{n,m+1} & \cdots & D_{nn} \end{array}\right]$$

Note that I am suppressing some information by simply writing D_{11} instead of $D_{11}(g)$ for each component. Operating on a basis vector v_i, with respect to the ordered basis, yields the vector

$$D(g)v_i=\begin{bmatrix} D_{i,1} \\ \vdots \\ D_{i,m} \\ D_{i,m+1} \\ \vdots \\ D_{i,n} \end{bmatrix}.$$

However, for the basis vectors v_i where $1 \le i \le m$ we know that $D(g)v_i \in U_m$ and thus the components of $D(g)v_i$ zero outside of the subspace. We now have a representation of the form

$$\left[\begin{array}{c|c} D^{(1)}(g) & X(g) \\ \hline 0 & D^{(2)}(g) \end{array}\right].$$

Apply the homomorphic property $D(g_1)=D(g_2)D(g_3)$ for some $g_1=g_2g_3 \in G$, we find that

$$\begin{aligned} D(g_1) &= \left[\begin{array}{c|c} D^{(1)}(g_2) & X(g_2) \\ \hline 0 & D^{(2)}(g_2) \end{array}\right] \cdot \left[\begin{array}{c|c} D^{(1)}(g_3) & X(g_3) \\ \hline 0 & D^{(2)}(g_3) \end{array}\right] \\ &= \left[\begin{array}{c|c} D^{(1)}(g_2)D^{(1)}(g_3) & X(g_2)D^{(1)}(g_3)+D^{(2)}(g_2)X(g_3) \\ \hline 0 & D^{(2)}(g_2)D^{(2)}(g_3) \end{array}\right]. \\ &= \left[\begin{array}{c|c} D^{(1)}(g_1) & X(g_1) \\ \hline 0 & D^{(2)}(g_1) \end{array}\right] \end{aligned}$$

From this we see that both $D^{(1)}$ and $D^{(2)}$ also give us representations of the group G because they obey the homomorphic property. But while $D^{(1)}$ seems to be invariant because it doesn't 'leak' into the other subspace (the lower left hand partition of the matrix is all zeros), $D^{(2)}$ does leak out of it's subspace and is thus not invariant.

We can extended this argument by decomposing V_n into its G-invariant subspace $U_m^{(1)}$ and $U_m^{(1)}$'s *orthogonal complement* $U_{n-m}^{(2)}$ which is also a G-invariant subspace. Thus

$V_n = U_m^{(1)} \oplus U_{n-m}^{(2)}$. If $\beta = \{v_1, \ldots, v_m\}$ is an ordered basis for $U_m^{(1)}$ and $\delta = \{v_{m+1}, \ldots, v_n\}$ is an ordered basis for $U_{n-m}^{(2)}$, then $\gamma = \{v_1, \ldots, v_m, v_{m+1}, \ldots, v_n\}$ is ordered basis for V_n. Extending the same argument as above, we find that we now have a representation of the form

$$D(g) = \left[\begin{array}{c|c} D^{(1)}(g) & 0 \\ \hline 0 & D^{(2)}(g) \end{array}\right].$$

At this stage it should now be clear that both $D^{(1)}$ and $D^{(2)}$ are invariant.

So what would happen if we continued to decompose the representations $D^{(i)}$ using the same method until it can be reduced no more? The representation will be in block diagonal form and this leads us to the concept of irreducible representations.

Definition A representation $d : G \to GL(V_n, C)$ is considered *irreducible* if there exists no non-zero proper subspace of V_n invariant under G.

Theorem 1.2 Every representation is a direct sum of irreducible representations.

Proof Through the extended proof of the last theorem this should be a pretty logical result. Proof of this can shown by induction on the dimension of V_n.

1.4 Direct Product

1979 Nobel Prize winner Steven Weinberg wrote, "The universe is an enormous direct product of representation of symmetry groups." Here we examine the direct product and its representations.

Definition Given a finite collection of groups G_i, then the direct product is $G_1 \otimes G_2 \otimes \ldots \otimes G_n = \{(g_1, g_2, \ldots, g_n) | g_i \in G_i\}$. We define the binary operation by $(g_1, g_2, \ldots, g_n)(g_1', g_2', \ldots, g_n') = (g_1 g_1', g_2 g_2', \ldots, g_n g_n')$.

Example Consider the group $C_2 \otimes C_3$ where $\langle a \rangle = C_2$ and $\langle b \rangle = C_3$. The group elements are thus $(e,e), (e,b), (e,b^2), (a,e), (a,b)(a,b^2)$.

Definition Let $d^{(1)} : G \to GL(V_n, C)$ and $d^{(2)} : G \to GL(U_k, C)$ be two representations of G with basis $\beta = \{\vec{v}_1, \ldots \vec{v}_n\}$ and $\gamma = \{\vec{u}_1, \ldots \vec{u}_k\}$. We thus have that

$$D^{(1)}(g)\vec{v}_j = \sum_i \vec{v}_i D_{ij}(g) \text{ and } D^{(2)}(g)\vec{u}_l = \sum_k \vec{u}_k D_{kl}(g)$$

We now define the direct product to be

$$D^{(1\times 2)}(g)\vec{v}_j \vec{u}_l \equiv \left[D^{(1)}(g)\vec{v}_j\right]\left[D^{(2)}(g)\vec{u}_l\right] = \sum_{i,k} \vec{v}_j \vec{u}_l D_{ij}^{(1)}(g) D_{kl}^{(2)}(g).$$

This definition does still satisfy the homomorphic property, you can check this easily. The above definition has so many indices that it's a little daunting. So note that this is equivalent to saying

$$D^{(1\times 2)}(g)=\begin{bmatrix} D_{11}^{(1)}D^{(2)} & D_{12}^{(1)}D^{(2)} \\ D_{21}^{(1)}D^{(2)} & D_{22}^{(1)}D^{(2)} \end{bmatrix}=\begin{bmatrix} D_{11}^{(1)}D_{11}^{(2)} & D_{11}^{(1)}D_{12}^{(2)} & D_{12}^{(1)}D_{11}^{(2)} & D_{12}^{(1)}D_{12}^{(2)} \\ D_{11}^{(1)}D_{21}^{(2)} & D_{11}^{(1)}D_{22}^{(2)} & D_{12}^{(1)}D_{21}^{(2)} & D_{12}^{(1)}D_{22}^{(2)} \\ D_{21}^{(1)}D_{11}^{(2)} & D_{21}^{(1)}D_{12}^{(2)} & D_{22}^{(1)}D_{11}^{(2)} & D_{22}^{(1)}D_{12}^{(2)} \\ D_{21}^{(1)}D_{21}^{(2)} & D_{21}^{(1)}D_{22}^{(2)} & D_{22}^{(1)}D_{21}^{(2)} & D_{22}^{(1)}D_{22}^{(2)} \end{bmatrix}$$

if both $D^{(1)}$ and $D^{(2)}$ are two dimensional representations. This is rather easy to calculate.

The direct product of two irreducible representations is not necessary irreducible. The direct product representation may be the direct sum of irreducible representations. To determine exactly how the direct product of a representation decomposes into direct sums, we need a few more tools that we will develop studying characters.

§ Chapter 2: Characters

2.1 Introduction

Having established the basic of representations in the last chapter, we now need a few more tools in order to work with them. Ultimately we would like to be able to take the direct product of two representations and then decompose the direct product into irreducible sums. Solving this problem is equivalent to finding a 'good' set of basis vectors that block diagonalize the matrices of the representation. This is exactly the problem that must be solved in order to find Clebsch-Gordon coefficients as we will see later.

This chapter will be presented in a rather unconventional manner. We will begin with a quick review and by presenting the major results of the chapter. This will help to provide structure and motivation for the subsequent proofs and derivation of the results.

2.2 The Tools

Recall the definition of a conjugacy class.

Definition 2.1 Let G be a group. Define $Cl(a)=\{xax^{-1} : x \in G\}$ and $a \in G$ as the conjugacy class of a. The conjugacy class partitions the group G.

Example 2.2
$D_3 = \{x^2 = e, y^2 = e, yx = x^2 y\}$ has three conjugacy classes:

$\{e\}$
$\{x, x^2\}$
$\{y, yx, yx^2\}$

Definition 2.2 The *character* of a representation $D^{(i)}$ of G is defined as $\chi^{(i)}(g) = \mathrm{Tr}\left(D^{(i)}(g)\right)$. The ordered collection of the characters of the elements for a representation is denoted by $\chi^{(i)} = \{\mathrm{Tr}(g) | g \in G\}$. It may also be convenient to denote this vector as a ket $\left|\chi^{(i)}\right\rangle$.

Theorem 2.3

i. Two equivalent representations have the same character.
ii. Elements in the same conjugacy class have the same character.
iii. If a representation is unitary, then $\chi\left(g^{-1}\right) = \chi(g)^*$ (complex conjugate).

Proof

i. Recall from linear algebra that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$. Let $D^{(1)}$ and $D^{(2)}$ be two equivalent representations and S the basis-transformation matrix such that $D^{(1)} = SD^{(2)}S^{-1}$. Notice that $Tr\left(D^{(1)}\right) = Tr\left(SD^{(2)}S^{-1}\right) = Tr\left(S^{-1}SD^{(2)}\right) = Tr\left(D^{(2)}\right)$. In words, two equivalent representations have the same character. This is a great result because it means that when we use characters, we do so without having to choose a particular basis for our representation.
ii. Elements in the same conjugacy class of a group have the same character. Let's assume that a and b are conjugates given by $b = xax^{-1}$. Then
$$Tr(D(b)) = Tr\left(D(x)D(a)D\left(x^{-1}\right)\right) = Tr\left(D(x)D\left(x^{-1}\right)D(a)\right)$$
$$= Tr\left(D\left(xx^{-1}\right)D(a)\right) = Tr(D(e)D(a)) = Tr(D(a))$$
and thus all elements in a conjugacy class have the same character.
iii. If D is unitary then $D^{-1} = D^{\dagger}$ (the conjugate transpose).
$$\chi\left(g^{-1}\right) = Tr\left(D(g)^{-1}\right) = Tr\left(D(g)^{\dagger}\right) = \chi(g)^*$$

With this, we now present the major results of this chapter.

The Tools

1. The number of irreducible representations of a group G is equal to the number of its conjugacy classes, c.
2. Let n^i be the dimension of the irreducible representation D^i. Then $\sum_i n_i^2 = |G|$. In words, the sum of the squares of the dimensions of the irreducible representations is equal to the size of the group.

3. $\frac{1}{|G|}\sum_{g\in G}\chi^{(i)}(g)^*\chi^{(j)}(g)=\delta_{ij}$. We denote this inner product with $\langle\chi^{(i)}|\chi^{(j)}\rangle=\delta_{ij}$. Orthogonality of character for representations i and j.
4. The character of a representation can be expressed as a linear combination of the characters of the irreducible representations of a group. This is because the characters $|\chi^{(i)}\rangle$ span all of *group space.*

These are the tools necessary to decompose a representation into its irreducible parts. If a representation D has character $|\chi\rangle$ and the irreducible representations $D^{(i)}$ have characters $|\chi^{(i)}\rangle$, then we simply use the inner product. Thus, $|\chi\rangle=\sum_i\langle\chi|\chi^{(i)}\rangle|\chi^{(i)}\rangle$.

If you do not wish to see the derivation of the above results, then skip to the last section of the chapter and start looking at some examples.

1.3 Schur's Lemmas

Lemma Let D be an irreducible representation $D:G\to GL(V_n,C)$. If $A:V_n\to V_n$ is a linear transformation that commutes with D, $D(g)A=AD(g)\ \forall g\in G$, then A is a scalar.

Proof Because we are working over the algebraically close field of the complex numbers, we know that every linear transformation has at least one eigenvector and corresponding eigenvalue. We let $\vec{a}$ be an eigenvector of A with corresponding eigenvalue λ.

$$A\vec{a}=\lambda\vec{a}$$

We can now say that,

$$A\big(D(g)\vec{a}\big)=D(g)A\vec{a}=D(g)\lambda\vec{a}=\lambda\big(D(g)\vec{a}\big)$$

In other words, we see that $D(g)\vec{a}$ is also an eigenvector of A with eigenvalue of λ. Actually, $D(g)\vec{a}$ could be more than one eigenvector because we're looking at all $g\in G$. Additionally, this set of eigenvectors also forms a subspace $U_m\subseteq V_n$ that must be group invariant (because we used the group to build it). Because we assumed D is irreducible, we are left with two choices, either $U_m=V_n$ or $U_m=\{0\}$. The latter case is quickly ruled out because we know that the subspace contains at least one eigenvector, thus we conclude that $U_m=V_n$. This implies that $A\vec{a}=\lambda\vec{a}$ for all $\vec{a}\in V_n$. Thus $A=\lambda I_n$.

Lemma Let $D^{(1)}:G\to GL(V_n,C)$ and $D^{(2)}:G\to GL(W_m,C)$ be two inequivalent representations and $B:V_n\to W_m$ a mapping between the two vector spaces. If $BD^{(1)}(g)=D^{(2)}(g)B\ \forall g\in G$, then $B=0$.

Proof

Case 1 $n<m$**:** Let $\vec{v}\in V_n$ be arbitrary, then $B\big(D^{(1)}(g)\vec{v}\big)=\big(D^{(2)}(g)(B\vec{v})\big)$. This is just a fancy way of saying that $\big(D^{(2)}(g)(B\vec{v})\big)\in \mathrm{Im}(B)$ for all $g\in G$. But of course, we know

from the definition $B:V_n \to W_m$ that $\mathrm{Im}(B) \subseteq W_m$ and that the $\mathrm{Im}(B)$ is group invariant (created by $D^{(2)}$). We again invoke the irreducibility of the representation $D^{(2)}$, and thus either $\mathrm{Im}(B)=\{0\}$ or $\mathrm{Im}(B)=W_m$. However, the dimension of the image cannot be larger than the range, thus $\dim(\mathrm{Im}(A)) \le n$ which says that $m \le n$. We have reached a contradiction, and thus it must be true that $B=\{0\}$ because $\mathrm{Im}(B)=\{0\}$.

Case 2 $n > m$**:** We consider now the kernel of $B:V_n \to W_m$ by examining the relation $B\left(D^{(1)}(g)\vec{v}\right)=\left(D^{(2)}(g)(B\vec{v})\right)$ just as we did before, but instead we use the vectors $\vec{k}$ such that $B\vec{k}=\vec{0}$. Thus we now have that $B\left(D^{(1)}(g)\vec{k}\right)=\left(D^{(2)}(g)\left(B\vec{k}\right)\right)=D^{(2)}(g)\vec{0}=\vec{0}$. Thus $D^{(1)}(g)\vec{k} \in \ker(B)$ for all $g \in G$ and the kernel of B is a group invariant subspace. Invoking irreducibility, we know that either $\ker(B)=\{\vec{0}\}$ or $\ker(B)=V_n$. However, we know that there must be something in the kernel B because the transformation B reduces dimensionality (range larger than the domain, $n > m$) and therefore $\ker(B)=V_n$. So if the kernel B is the whole vector space, then $B=0$.

Case 3 $n = m$**:** Consider the same argument where we know that $\ker(B)=\{\vec{0}\}$ or $\ker(B)=V_n$. This time if $\ker(B)=\{\vec{0}\}$, then B is one-to-one and thus invertible. This would imply that we could write $D^{(2)}(g)=B^{-1}D^{(1)}(g)B$ and thus $D^{(2)}(g)=D^{(1)}(g)$, a contradiction to our assumption that these are inequivalent representations. We have thus shown for the third and final time that $B=0$.

1.4 Orthogonality

Let $D^{(\nu)}:G \to GL(V_n,C)$ and $D^{(\mu)}:G \to GL(W_m,C)$ be two representations and $A:V_n \to W_m$ a mapping between the two vector spaces. We now define the operator B,

$$B=\sum_g D^{(\mu)}(g)AD^{(\nu)}\left(g^{-1}\right)$$

This is the same ordered sum that was used to define the character previously. Notice what we're trying to do here. The object is to find an orthogonality relation, B, between the different irreducible relations and we will do this by exploiting Schur's Lemmas. We're summing over all of the group elements because, of course, we need to know about the entire groups behavior. We would expect that if the irreducible representations are equivalent, that gg^{-1} will just give us the identity element e. The representation of e is, of course the identity, and we're summing $|G|$ times. So we might expect the sum, and therefore B, to collapse to the size of the group $|G|$ if the irreducible representations are equivalent (Schur's First Lemma) and the sum to be zero if they are inequivalent (Schur's Second Lemma). Let's show this now.

Consider now the elements h and h^{-1} of G with the representations $D^{(\mu)}(h)$ and $D^{(\nu)}\left(h^{-1}\right)$. Multiply the sum with these elements and we have that

$$D^{(\mu)}(h)BD^{(\nu)}\left(h^{-1}\right)=\sum_g D^{(\mu)}(h)D^{(\mu)}(g)AD^{(\nu)}\left(g^{-1}\right)D^{(\nu)}\left(h^{-1}\right)$$

We now use the homomorphic property and find that

$$D^{(\mu)}(h)BD^{(\nu)}\left(h^{-1}\right)=\sum_g D^{(\mu)}(hg)AD^{(\nu)}\left(g^{-1}h^{-1}\right)$$

Recall that $g^{-1}h^{-1}=(hg)^{-1}$ and thus

$$D^{(\mu)}(h)BD^{(\nu)}\left(h^{-1}\right)=\sum_g D^{(\mu)}(hg)AD^{(\nu)}\left((hg)^{-1}\right)$$

But the right side of the equation is just equal to B and we therefore have the relation

$$D^{(\mu)}(h)BD^{(\nu)}\left(h^{-1}\right)=B$$

Now it should be clear that we can use Schur's Lemmas. B is zero if the irreducible representations are inequivalent, but is some scalar times the identity if they are equal. We write this as

$$\sum_g D^{(\mu)}(g)AD^{(\nu)}\left(g^{-1}\right)=\lambda I_n\delta^{\mu\nu}$$

We can actually figure out what λ is by taking the trace of both sides. We find that

$$\sum_g Tr\left(D^{(\mu)}(g)AD^{(\mu)}\left(g^{-1}\right)\right)=Tr(\lambda I_n)$$

$$\sum_g Tr\left(D^{(\mu)}(g)D^{(\mu)}\left(g^{-1}\right)A\right)=\lambda n$$

$$\sum_g Tr(A)=\lambda n$$

$$|G|Tr(A)=\lambda n$$

From which it is clear that $\lambda=\frac{|G|}{n}Tr(A)$. We now have the following relation

$$\sum_g D^{(\mu)}(g)AD^{(\nu)}\left(g^{-1}\right)=\frac{|G|}{n}Tr(A)I_n\delta^{\mu\nu} \quad \textbf{(2.1)}$$

To proceed any further we are going to have to write the sums with the representations in terms of their components. However, before we do that it is useful to review summation of matrices.

Review
Let $D^1(g)=D^1_{ij}(g)$ and $D^2(g)=D^2_{kl}(g)$ be two matrices whose product, $D^3(g)=D^1(g)D^2(g)$, can be written as $D^3(g)=D^3_{mn}(g)$. The product $D^3(g)=D^1(g)D^2(g)$, written with components, reads $D^3_{ik}(g)=\sum_j D^1_{ij}(g)D^2_{jk}(g)$.

Let $D^\mu(g)=D^\mu_{i_1 j_1}(g)$, $D^\nu(g)=D^\nu_{i_2 j_2}(g)$ and $A=A_{i_3 j_3}$. The left side of equation 2.1 becomes

$$\sum_{g,j_1,j_2} D^{\mu}_{i_1 j_1}(g) A_{j_1 j_2} D^{\nu}_{j_2 i_2}\left(g^{-1}\right)$$

A fancy way of writing the trace would be to say $Tr(A) = \sum_{j_1,j_2} \delta_{j_1 j_2} A_{j_1 j_2}$. Additionally, the identity I_n can be written as $I_n = \delta_{i_1 i_2}$. Thus, equation 2.1 now becomes

$$\sum_{g,j_1,j_2} D^{\mu}_{i_1 j_1}(g) A_{j_1 j_2} D^{\nu}_{j_2 i_2}\left(g^{-1}\right) = \frac{|G|}{n} \delta^{\mu\nu} \sum_{j_1,j_2} \delta_{i_1 i_2} \delta_{j_1 j_2} A_{j_1 j_2}$$

Equating the coefficients of $A_{j_1 j_2}$ two of the sums are gone and we are left with

$$\sum_{g} D^{\mu}_{i_1 j_1}(g) D^{\nu}_{j_2 i_2}\left(g^{-1}\right) = \frac{|G|}{n} \delta^{\mu\nu} \delta_{i_1 i_2} \delta_{j_1 j_2} \quad \textbf{(2.2)}$$

This is what many authors refer to as "The Fundamental Orthogonality Theorem" or sometimes even "The Great Orthogonality Theorem." Let's take a look and review what this theorem tells us about irreducible representations. The first delta function $\delta^{\mu\nu}$ says that we'd better be dealing with the same representation or else we're going to get zero. The next two delta functions $\delta_{i_1 i_2} \delta_{j_1 j_2}$ tell us that the representation of g and its inverse had better be the transpose of each other, otherwise we're going to get zero.

To find the orthogonality of the characters we just need to take the appropriate traces. We want the trace of the two representations, so we can do this by simply multiplying by the delta functions $\delta_{i_1 j_1} \delta_{j_2 i_2}$. We thus have the following mess

$$\sum_{g} \delta_{i_1 j_1} \delta_{j_2 i_2} D^{\mu}_{i_1 j_1}(g) D^{\nu}_{j_2 i_2}\left(g^{-1}\right) = \frac{|G|}{n} \delta^{\mu\nu} \delta_{i_1 i_2} \delta_{j_1 j_2} \delta_{i_1 j_1} \delta_{j_2 i_2}$$

The delta functions $\delta_{j_1 j_2} \delta_{i_1 j_1} \delta_{j_2 i_2}$ collapse to $\delta_{i_1 i_2}$ and we can also write the traces in terms of characters, thus

$$\sum_{g} \chi^{\mu}(g) \chi^{\nu}\left(g^{-1}\right) = \frac{|G|}{n} \delta^{\mu\nu} \delta_{i_1 i_2} \delta_{i_1 i_2}$$

However, $\delta_{i_1 i_2} \delta_{i_1 i_2}$ is simply $\delta_{i_1 i_1}$ which is just n. Recall also that $\chi^{\nu}\left(g^{-1}\right) = \chi^{\nu}(g)^*$. Thus,

$$\frac{1}{|G|} \sum_{g} \chi^{\mu}(g) \chi^{\nu}(g)^* = \delta^{\mu\nu} \quad \textbf{(2.3)}$$

This shows the orthogonality of characters. We therefore define the inner product

$$\left\langle \chi^{\mu} \middle| \chi^{\nu} \right\rangle \equiv \frac{1}{|G|} \sum_{g} \chi^{\mu}(g) \chi^{\nu}(g)^* = \delta^{\mu\nu}$$

This is a very important result that makes dealing with representations a very simple task.

1.5 Number of Irreducible Representations

In addition to orthogonality of characters for each element, we can also show that the conjugacy classes characters are orthogonal. Let's denote the conjugacy classes by K_i with n_i elements – with a total of c classes. It should be pretty clear that we can rewrite the orthogonality theorem as

$$\frac{1}{|G|}\sum_{i=1}^{c} n_i \chi^{\mu}(K_i)\chi^{\nu}(K_j)^* = \delta_{ij}.$$

This implies that there can be at most c mutually orthogonal vectors (characters). But of course, this also just means that that there can be at most c irreducible representations. Let's denote the number of irreducible representation by n_r, thus $n_r \le c$.

Alternatively, it is also possible to rewrite the orthogonality theorem as

$$\frac{1}{|G|}\sum_{\mu} n_i \chi^{\mu}(K_i)\chi^{\mu}(K_j)^* = \delta_{ij}$$

Using the same argument as before, this result implies that $c \le n_r$. From which we conclude that the number of irreducible representations is equal to the number of conjugacy classes.

Theorem Let $D: G \to GL(V)$ be a representation with character $|\chi\rangle$ and let D_i be the irreducible representations with character $|\chi_i\rangle$. Then D decomposes in the direct sum of irreducible representations

$$D = m_1 D_1 \oplus \ldots \oplus m_c D_c$$

where $m_i = \langle \chi | \chi_i \rangle$ is the number of occurrences of D_i.

Proof Think characters!

The last item for us to show is that the sum of the squares of the dimensions of the irreducible representations is equal to the size of the group.

Theorem Let n^i be the dimension of the irreducible representation D^i. Then $\sum_i n_i^2 = |G|$.

Proof Ich habe kein Bock – will das nicht beweisen…

1.6 Examples

Example

Consider now the group $C_3 = (e, r, r^2)$. We know the group C_3 has three irreducible representations from fact (1) above. Exploiting fact (2) we know that $n_1^2 + n_2^2 + n_3^2 = 3$ where n_i is the dimension of the representation. The only integer solution to this equation is $1+1+1=3$ and we thus have three one-dimensional representations of the group. Inserting the trivial representation, the character table looks the table below.

C_3	e	r	r^2
$D^{(1)}$	1	1	1
$D^{(2)}$	1		
$D^{(3)}$	1		

It to find the remaining pieces we use the fact that the characters of 1-dimensional representations are the representations themselves. Thus we know that

$$\left(D^{(2)}(r)\right)^3 = 1$$
$$\Rightarrow D^{(2)}(r) = e^{i\theta} \text{ and } e^{i2\theta}$$
$$\text{where } \theta = \frac{2\pi}{3}.$$

C_3	e	r	r^2
$D^{(1)}$	1	1	1
$D^{(2)}$	1	$e^{i\theta}$	$e^{i2\theta}$
$D^{(3)}$	1	$e^{i2\theta}$	$e^{i\theta}$

At this point it is also important to notice that $e^{i2\theta} = e^{-i\theta}$ and thus we can again write the character table as

C_3	e	r	r^2
$D^{(1)}$	1	1	1
$D^{(2)}$	1	$e^{i\theta}$	$e^{i2\theta}$
$D^{(3)}$	1	$e^{-i\theta}$	$e^{-i2\theta}$

Now let's consider the three-dimensional Euclidean vector space $V = R^3$ and the standard basis. If we wish to consider C_3 as rotations about the z-axis, we already know how to write one such three dimensional representation, D^V.

$$D^V(e) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, D^V(r) = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, D^V(r^2) = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

From this we can we see that $\left|\chi^V\right\rangle = \{3,0,0\}$ with respect to the basis of $\beta = \{e, r, r^2\}$ in group space. Dotting this with the three basis vectors $\left|\chi^{(1)}\right\rangle, \left|\chi^{(2)}\right\rangle, \left|\chi^{(3)}\right\rangle$ it should be clear that $\left|\chi^V\right\rangle = \left|\chi^{(1)}\right\rangle + \left|\chi^{(2)}\right\rangle + \left|\chi^{(3)}\right\rangle$. From this we now know that $D^V = D^{(1)} \oplus D^{(2)} \oplus D^{(3)}$.

We now need to assign a basis vector to each dimension of each irreducible representation that makes up D^V. For example, knowing that we want the z-axis fixed under this group we can assign the basis $\gamma = \{\phi_1(x,y,z) = z, \phi_2(x,y,z) = x, \phi_3(x,y,z) = y\}$ for $D^{(1)}, D^{(2)}, D^{(3)}$, respectively. We now observe what a group element does to each of these vectors.

$$D^{(1)}(r)\phi_1(x,y,z)=1\cdot z=z$$
$$\Rightarrow z'\to z$$
$$D^{(2)}(r)\phi_2(x,y,z)=e^{i\theta}x$$
$$\Rightarrow x'\to e^{i\theta}x$$
$$D^{(3)}(r)\phi_3(x,y,z)=e^{-i\theta}y$$
$$\Rightarrow y'\to y$$

In the standard basis of $\left\{x=\begin{bmatrix}1\\0\\0\end{bmatrix},y=\begin{bmatrix}0\\1\\0\end{bmatrix},z=\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$, we now have that

$$D^V(r)=\begin{bmatrix}e^{i\theta} & 0 & 0\\ 0 & e^{-i\theta} & 0\\ 0 & 0 & 1\end{bmatrix}$$

and similar construction will yield the other two elements.

Something a little bit more fancy might be to use the basis that will give us the rotation matrices we are used to seeing. Out of thin air we find and then decide to use the basis $\gamma=\{z,x+iy,x-iy\}$. Now we have

$$D^{(1)}(r)\phi_1(x,y,z)=1\cdot z=z$$
$$\Rightarrow z'\to z$$
$$D^{(2)}(r)\phi_2(x,y,z)=e^{i\theta}(x+iy)$$
$$=(\cos\theta+i\sin\theta)(x+iy)$$
$$=(x\cos\theta-y\sin\theta)+i(x\sin\theta+y\cos\theta)$$

and the last term is in the form $x+iy$ denoting that

$$x'\to x\cos\theta-y\sin\theta$$
$$y'\to x\sin\theta+\cos\theta$$

The basis we chose for $D^{(3)}$ yields the same result as $D^{(2)}$. We have determined the components of $D^V(r)$ in this new basis, namely,

$$D^V(r)=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}.$$

Example 4

It can be shown that the character table for C_n where ξ^n denotes the nth root of 1, is the following:

C_n	e	r	r^2	$\ldots$	r^{n-1}
$\chi^{(1)}$	1	1	1	$\ldots$	1
$\chi^{(2)}$	1	ξ	ξ^2	$\ldots$	ξ^{n-1}

$\chi^{(3)}$	1	ξ^{2}	ξ^{3}	$\cdots$	$\xi^{2(n-1)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\chi^{(n)}$	1	ξ^{n-1}	$\xi^{2(n-1)}$	$\cdots$	$\xi^{(n-1)^2}$

So now if you'll notice, the character in the n[th] representation is the complex conjugate of the character in the 2[nd] representation. It turns out that the direct sum of $D^{(1)} \oplus D^{(2)} \oplus D^{(n)}$ in the basis $\{z, x+iy, x-iy\}$ will give you the following representation,

$$D^{V}\left(r^{n}\right)=\begin{bmatrix} \cos(n\theta) & -\sin(n\theta) & 0 \\ \sin(n\theta) & \cos(n\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is, of course, pretty dang cool and should be relatively clear given the last example we worked out in more detail.

§ Chapter 3: Infinite Rotational Group

3.1 Introduction

The object now is to examine the three-dimensional rotation group, commonly denoted by $SO(3)$ -- the **S**pecial **O**rthogonal group of three dimensions and determinant 1. The idea here is to look at this group the same way we looked at the other simpler groups, such as C_3, by finding the representations and their characters of the group. Ultimately, because this group is infinite, it will prove to be much more difficult to achieve our goals. However, there are many rewards that we will pick up along the way.

3.2 Generators

Often groups are defined in terms of their generators such as the dihedral-3 group $D_3 = \left\langle x, y \middle| x^3 = e, y^2 = e \right\rangle = \left\{e, x, x^2, y, yx, yx^2\right\}$ where x and y are the generators of the group. We will now try to find the generators of $SO(3)$. We use the notation $R(\alpha,\xi)$ to denote the rotation by α about the ξ axis. I_ξ represents an infinitesimally small rotation about the ξ axis. From basic calculus, we know that

$$I_\xi = \lim_{x\to 0}\left(\frac{R(\alpha,\xi)-R(0,\xi)}{\alpha}\right) = \left.\frac{dR(\alpha,\xi)}{d\alpha}\right|_{\alpha=0} . \quad \textbf{(2.1)}$$

This is great, but there are two things to note at this point. First, as a matter of convention and convenience we will denote the above limit by iI_ξ instead of I_ξ. Second, note that $R(0,\xi)$ is actually just the identity matrix, i.e., we don't rotate at all. We now have

$$iI_\xi = \lim_{x\to 0}\left(\frac{R(\alpha,\xi)-I}{\alpha}\right)$$

where I represents the identity matrix. Rearranging the above equation, for small α we can write

$$R(\alpha,\xi) \approx I + iaI_\xi .$$

Here's a little bit of trickery: to rotate by angle α we now rotate by $\frac{\alpha}{n}$ a total of n times. Hence,

$$R(\alpha,\xi) = \left(R\left(\frac{\alpha}{n},\xi\right)\right)^n \approx \left(I + \frac{\alpha}{n} iI_\xi\right)^n .$$

If we take this limit we no longer have an approximation.

$$R(\alpha,\xi) = \lim_{n\to\infty}\left(I + \frac{\alpha iI_\xi}{n}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha iI_\xi)^n}{n!} = e^{\alpha iI_\xi}$$

The last step follows from the definition of the exponential function from calculus. Great. So now we have a way rotation about an axis in terms of the operator I_ξ. But just what is this operator and what does it look like in the standard basis for R^3? As you intuitively suspect, we can represent I_ξ in terms a linear combination of I_x, I_y and I_z. The rotations about the x, y and z axis can be represented by

$$R(\alpha,z) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R(\alpha,y) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

$$R(\alpha,x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}$$

From equation 2.1 we can find the infinitesimal rotations for this representation

$$iI_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, iI_y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, iI_z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$I_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, I_y = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, I_z = \begin{bmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now we can write I_ξ in terms a linear combination of I_x, I_y and I_z, $I_\xi = aI_x + bI_y + cI_z$. From inspection of the coordinate system we can determine the missing coefficients (Heine, 53). If θ is the angle of the $\vec{\xi}$ vector from the z-axis down to the x-y plane and ϕ

is it angle from the x-axis, then $I_\xi = \sin\theta\cos\phi I_x + \sin\theta\sin\phi I_y + \cos\theta I_z$. We have now completed the first of our goals. We can write any rotation in R^3 in terms of the three generators I_x, I_y and I_z.

3.3 Commutation Relations

The next thing that is useful to consider is the commutations between the three generators. It should be pretty intuitively clear the three operators don't commute. The simplest method for determining the commutation relations is to just calculate them using the representations from above. We could look at the general case, but this representation works just fine too.

$$\begin{aligned}
&\left[I_x, I_y\right] = I_x I_y - I_y I_x \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= iI_z
\end{aligned}$$

If we use this same procedure we can find the two other relations,

$$\begin{aligned}
\left[I_x, I_y\right] &= iI_z \\
\left[I_x, I_z\right] &= -iI_y \\
\left[I_y, I_z\right] &= iI_x
\end{aligned}$$

These relations turn out to be somewhat useful later. Otherwise they are said to define an algebra, whatever that means…

3.4 The Irreducible Representations

If you recall from the last chapter, when we were trying to find the irreducible representations of some group we looked for invariant subspaces of the group. We looked for subspaces, V_n, that contained no proper group invariant subspaces. A set of basis vectors of V_n could be used to make that particular irreducible representation of that group. Our approach is going to be the same here. We are going to search for the basis vectors of subspace invariant under the full rotation group. Because any element of the full rotation group can be represented by a linear combination of the three generators, we will use the generators to find the invariant subspaces.

A very nice set of vectors to work with, are the eigenvectors of the rotation operators. However, from the commutation relations, we know that the generators cannot have simultaneous eigenvectors. By convention we will work with the eigenvectors of the I_z operator. Let's start by working in some finite n-dimensional subspace V_n that we assume is group invariant. We can now assume that I_z has a least one eigenvector in V_n, let's denote it by $I_z|m\rangle = m|m\rangle$.

Let's see if we can find the other eigenvectors of I_z. The recommend way of doing this is to create an operator, call it S, that moves from the one eigenvector that we know exists, $|m\rangle$, to the other eigenvectors I_z. Symbolically we want $\{S|I_z(S|m\rangle) = \lambda(S|m\rangle)\}$. Amazingly, just by examining the commutation relations we can determine what S actually has to be. Using $|m\rangle$ as our test function,

$$
\begin{aligned}
&I_zS|m\rangle - SI_z|m\rangle \\
&= \lambda S|m\rangle - mS|m\rangle \\
&= (\lambda - m)S|m\rangle
\end{aligned}
$$

From which we see that $[I_z, S] = kS$ for some k. Of course, everything in the group can be written as a linear combination of the generators, so we can write $S = aI_x + bI_y + cI_z$. Let's try the commutation relation one more time.

$$
\begin{aligned}
[I_z, S] &= I_zS - SI_z \\
&= I_z(aI_x + bI_y + cI_z) - (aI_x + bI_y + cI_z)I_z \\
&= aI_zI_x + bI_zI_y + cI_zI_z - aI_xI_z - bI_yI_z - cI_zI_z \\
&= a(I_zI_x - I_xI_z) + b(I_zI_y - I_yI_z) \\
&= a(iI_y) + b(-iI_x) \\
&= -ibI_x + iaI_y
\end{aligned}
$$

But since we know from above that $[I_z, S] = k(aI_x + bI_y + cI_z)$, we can equate the coefficients. Doing this we find that $c = 0$, $a = \frac{1}{k}(-ib)$ and $a = k(-ib)$. The only values of k that satisfy the relation $k = \frac{1}{k}$ are $k = \pm 1$. From this we now see that $S = a(I_x \pm iI_y)$. Letting $a = 1$ we now denote the two S operators as $I_+ = I_x + iI_y$ and $I_- = I_x - iI_y$ with commutation relations $[I_z, I_\pm] = \pm I_\pm$.

Let's examine what $I_\pm$ do to the eigenvector $|m\rangle$ of I_z.

$$
\begin{aligned}
I_zI_\pm|m\rangle &= I_zI_\pm|m\rangle \\
I_zI_\pm|m\rangle &= I_zI_\pm|m\rangle + I_\pm I_z|m\rangle - I_\pm I_z|m\rangle \\
I_zI_\pm|m\rangle &= [I_z, I_\pm]|m\rangle + I_\pm I_z|m\rangle \\
I_zI_\pm|m\rangle &= \pm I_\pm|m\rangle + mI_\pm|m\rangle \\
I_zI_\pm|m\rangle &= (m \pm 1)I_\pm|m\rangle
\end{aligned}
$$

So it seems that I_+ moves the eigenvector $|m\rangle$ to an eigenvector with an eigenvalue 1 higher. Let's write this as $I_+|m\rangle = c_m|m+1\rangle$ where the constant c_m is yet to be determined. Similarly, I_- lowers the eigenvector. This is a little bit of a problem because we want to deal with finite vector spaces, so we can just arbitrarily say that the $|m = j\rangle$

vector is the last one and thus $I_+|j\rangle = 0$. At this point it's thus probably good to change our notation a little and write our vectors as $|j,m\rangle$ so we know that we can only raise m up to j before we're done and hit the wall.

We need to take care of one thing that will soon be useful. We need to determine $[I_+,I_-]$.

$$
\begin{aligned}
[I_+,I_-] &= I_+I_- - I_-I_+ \\
&= (I_x + iI_y)(I_x - iI_y) - (I_x - iI_y)(I_x + iI_y) \\
&= I_xI_x - iI_xI_y + iI_yI_x + I_yI_y - I_xI_x - iI_xI_y + iI_yI_x - I_yI_y \\
&= -2i[I_x,I_y] \\
&= 2I_z
\end{aligned}
$$

Given that $I_+|m\rangle = c_m|m+1\rangle$, we will similarly define $I_-|m\rangle = d_m|m-1\rangle$. Let's see how c_m and d_{m+1} relate.

$$
\begin{aligned}
&\langle m-1|I_+|m\rangle = \langle m|c_m|m\rangle = c_m \\
&\langle m|I_-|m+1\rangle = \langle m|d_{m+1}|m\rangle = d_{m+1} \\
&\Rightarrow c_m = d_{m+1}
\end{aligned}
$$

Now we're set to determine c_m.

$$
\begin{aligned}
I_+|m-1\rangle &= I_+\left(\frac{1}{d_m}I_-|m\rangle\right) = \frac{1}{d_m}I_+I_-|m\rangle \\
&= \frac{1}{d_m}(I_-I_+ + 2I_z)|m\rangle \\
&= \frac{1}{d_m}(I_-I_+|m\rangle + 2I_z|m\rangle) \\
&= \frac{1}{d_m}(d_{m+1}c_m + 2m)|m\rangle
\end{aligned}
$$

but we also know that $I_+|m-1\rangle = c_{m-1}|m\rangle$. Setting the component equal we see that $\frac{1}{d_m}(d_{m+1}c_m + 2m) = c_{m-1}$.

$$
\begin{aligned}
&\frac{1}{d_m}(d_{m+1}c_m + 2m) = c_{m-1} \\
&\Rightarrow \frac{1}{c_{m-1}}\left({c_m}^2 + 2m\right) = c_{m-1} \\
&\Rightarrow {c_m}^2 + 2m = {c_{m-1}}^2
\end{aligned}
$$

We're almost there! We now have a recursive equation that establishes the missing coefficients. Solutions to this type of equation are often solved in numerical analysis books, see Stegner's *Diskrete Strukturen* for example of this. We start by substituting ${c_m}^2 = b_m$ and reducing it to a linear equation. Now,

$$b_{m-1} = b_m + 2m$$
$$\Rightarrow b_{m-1} - b_m - 2m = 0$$
$$\Rightarrow b_m - b_{m+1} - 2(m+1) = 0$$

The difference of these last two equations gets rid of that pesky m.

$$\begin{array}{l} b_{m-1} - b_m - 2m = 0 \\ \underline{-b_m + b_{m+1} + 2m + 2 = 0} \\ b_{m-1} - 2b_m + b_{m+1} + 2 = 0 \end{array}$$

Now we just need to get rid of the 2, so we use the same process one more time.

$$\begin{array}{l} b_{m-1} - 2b_m + b_{m+1} + 2 = 0 \\ \underline{-b_m + 2b_{m+1} - b_{m+2} - 2 = 0} \\ -b_{m+2} + 3b_{m+1} - 3b_m + b_{m-1} = 0 \end{array}$$

Finally we can write and solve the characteristic equation for the recursion.

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$$
$$\Rightarrow (\lambda - 1)^3 = 0$$

Thus closed form solutions to the recursion take the form

$$b_m = \alpha m^2 1^m + \beta m 1^m + \gamma 1^m$$

or simply,

$$b_m = \alpha m^2 + \beta m + \gamma .$$

with still to be determined coefficients α, β and γ. To determine these coefficients we can plug in the first three iterations of the recursions. We know that we $b_j = 0$ because that's to be the end of the ladder. Thus,

$$b_{j-1} = b_j + 2j$$
$$\Rightarrow b_{j-1} = 2j$$

Similarly,

$$b_{j-2} = b_{j-1} + 2(j-1)$$
$$\Rightarrow b_{j-2} = 2j + 2j - 2$$
$$\Rightarrow b_{j-2} = 4j - 2$$

We can write that,

$$b_j = \alpha j^2 + \beta j + \gamma = 0$$
$$b_{j-1} = \alpha (j-1)^2 + \beta (j-1) + \gamma = 2j$$
$$b_{j-2} = \alpha (j-2)^2 + \beta (j-2) + \gamma = 4j - 2$$

Three equations and three unknowns can be easily solved. In matrix form,

$$\begin{bmatrix} j^2 & j & 1 \\ (j-1)^2 & j-1 & 1 \\ (j-2)^2 & j-2 & 1 \end{bmatrix} \cdot \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 2j \\ 4j-2 \end{bmatrix}$$

From which it easily determined that $\alpha = -1$, $\beta = -1$ and $\gamma = j(j+1)$. Thus,

$$b_m = -m^2 - m + j(j+1)$$
$$\Rightarrow b_m = j(j+1) - m(m+1)$$
$$\Rightarrow c_m = \sqrt{j(j+1) - m(m+1)}$$

We can use the exact same process to determine the coefficient for the I_- operator acting on an eigenvector of I_z. Our final results are thus,

$$I_z|j,m\rangle = m|j,m\rangle$$
$$I_+|j,m\rangle = \sqrt{j(j+1) - m(m+1)}|j,m+1\rangle$$
$$I_-|j,m\rangle = \sqrt{j(j+1) - m(m-1)}|j,m-1\rangle$$

From this we can see that not only does $I_+|j,j\rangle = 0$, but $I_-|j,-j\rangle = 0$. So it would seem that for some value of j, m can take on values ranging from j to $-j$. Thus for some value of j there are $2j+1$ eigenvectors of I_z. Because we're working in some n-dimensional vector space and n is a whole number, this places a limit on what values j can assume.

$$n = 2j+1$$
$$\Rightarrow j = \frac{n+1}{2}$$

So it seem that j can assume any half-integer value, $j = \frac{1}{2}, 1, \frac{3}{2}, \ldots$.

So what have we done? We have found what the irreducible representations of the rotation group look like using the eigenvectors of I_z as a basis. Right now we have everything in terms of I_z, I_+ and I_-, but because we know how $I_\pm$ relate to I_x and I_y, we can easily recover their form. Also remember that because we can write I_ξ in terms a linear combination of I_x, I_y and I_z, we have the irreducible representations of and arbitrary rotation I_ξ. Let's explicitly write out the first two irreducible representations for I_+, I_-, I_x, I_y and I_z,

For the $j = \frac{1}{2}$ representation we have two eigenvectors of I_z, $\left|\frac{1}{2},\frac{1}{2}\right\rangle$ and $\left|\frac{1}{2},-\frac{1}{2}\right\rangle$. With respect to those as an ordered basis we can now write,

$$I_z = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, I_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, I_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

We also know that $I_x = \frac{1}{2}(I_+ + I_-)$ and $I_y = \frac{1}{2i}(I_+ + I_-)$, thus

$$I_x = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, I_y = \frac{1}{2i}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

For the $j = 1$ representation we have the three eigenvectors $\{|1,1\rangle, |1,0\rangle, |1,-1\rangle\}$ forming basis. Thus,

$$I_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, I_+ = \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix}, I_+ = \begin{bmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}$$

$$I_x = \frac{1}{2}\begin{bmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{bmatrix}, I_y = I_x = \frac{1}{2i}\begin{bmatrix} 0 & \sqrt{2} & 0 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{bmatrix}$$

3.5 Characters of the irreducible representation

Just as we did in the last chapter, we can compute the characters of the irreducible representations. Finding the characters, as you recall, is very useful in determining which irreducible representations compose the reducible representation.