

## Problems 2.1 and 2.2

## An Algorithm to Solve the Rubik's Cube

## Notation:

- F is a 90 degree clockwise turn of the front face. B, R, L, U, D are similar moves for the Back, Right, Left, Up, and Down faces respectively.
- $F^{-1}$  is a 90 degree counterclockwise turn of the front face.
- $[FR] = FRF^{-1}R^{-1}$
- Let  $fu_o$  denote the cubelet that should be in the front upper position and let  $fu$  simply denote the front upper position. Ignore facelet orientation unless specified.
- Reorientations of the cube are often written as permutations of the faces in cycle notation. For example: (F,L,B,R) means fix the up and down faces and send the front face to the left, the left face to the back, the back face to the right, and the right face to the front.

## I. Solving the UP layer:

## A. Positioning edge cubelets:

- i. Find  $fu_o$  and move it to  $df$ .
  - a. If  $fu_o$  in UP layer then it is in one of the following positions:  $uf, ur, ub, ul$ .
    1. If it is in  $uf$ , use  $F^2$  to move it to the  $df$  position.
    2. If it is in  $ur$ , use  $R^2D^{-1}$  to move it to the  $df$  position.
    3. If it is in  $ub$ , use  $B^2D^2$  to move it to the  $df$  position.
    4. If it is in  $ul$ , use  $L^2D$  to move it to the  $df$  position.
  - b. If  $fu_o$  in Middle layer then it is in one of the following positions:  $fr, fl, bl, br$ .
    1. If it is in  $fr$ , use F to move it to the  $df$  position.
    2. If it is in  $fl$ , use  $F^{-1}$  to move it to the  $df$  position.
    3. If it is in  $bl$ , use  $U^2BU^{-2}D^2$  to move it to the  $df$  position.
    4. If it is in  $br$ , use  $U^2B^{-1}U^{-2}D^2$  to move it to the  $df$  position.
  - c. If  $fu_o$  in Down layer then it is in one of the following positions:  $df, dr, db, dl$ .
    1. If it is in  $df$ , continue to (ii).
    2. If it is in  $dr$ , use  $D^{-1}$  to move to  $df$  position.
    3. If it is in  $db$ , use  $D^2$  to move to  $df$  position.
    4. If it is in  $dl$ , use D to move to  $df$  position.
- ii. Properly position  $fu_o$  with proper orientation.
  - a. If the Up facelet of  $fu_o$  is facing down, use  $F^2$  to properly position the cubelet.
  - b. If the Up facelet of  $fu_o$  is facing front, use  $R^{-1}DRF^{-1}$  to properly position the cubelet.

iii. Reorient: (F,L,B,R), and repeat A. until all of the Up Edge cubelets are in position. Note they will be properly oriented.

B. Positioning UP corner cubelets.

i. Find  $fur_o$ .

a. If  $fur_o$  in UP layer then it is in one of the following positions:  $urb$ ,  $ulb$ ,  $ulf$ ,  $urf$ .

1. If  $fur_o$  is in  $urb$ , use  $UR^{-1}D^{-1}RU^{-1}$  to move to the  $dfl$  position.
2. If  $fur_o$  is in  $ulb$ , use  $U^2R^{-1}D^{-1}RU^{-2}$  to move to the  $dfl$  position.
3. If  $fur_o$  is in  $ulf$ , use  $U^{-1}R^{-1}D^{-1}RU$  to move to the  $dfl$  position.
4. If  $fur_o$  is in  $urf$ , use  $R^{-1}D^{-1}R$  to move to the  $dfl$  position.

b. If  $fur_o$  in Down layer then it is in one of the following positions:  $drb$ ,  $dlb$ ,  $drf$ ,  $dlf$ .

1. If  $fur_o$  is in  $drb$ , use  $D^2$  to move to the  $dfl$  position.
2. If  $fur_o$  is in  $dlb$ , use  $D$  to move to the  $dfl$  position.
3. If  $fur_o$  is in  $drf$ , use  $D^{-1}$  to move to the  $dfl$  position.
4. If  $fur_o$  is in  $dlf$ , continue to the next step.

ii. Position  $fur_o$  in proper position with proper orientation.

- a. If the Up facelet of  $fur_o$  is facing left, use  $R^{-1}DR$  to properly position the cubelet.
- b. If the Up facelet of  $fur_o$  is facing front, use  $D^2FD^{-1}F^{-1}$  to properly position the cubelet.
- c. If the Up facelet of  $fur_o$  is facing down, use  $(R^{-1}DR)(FDF^{-1})D^2(R^{-1}DR)$  to properly position the cubelet.

iii. Reorient: (F,L,B,R), and repeat B. until all of the Up corner cubelets are in position. Note they will be properly oriented.

II. Position Down corner cubelets.

A. Reorient: (U,D) (F,B).

B. One of three cases will occur:

i. It is possible using an element of  $\langle U \rangle$  to put only one corner cubelet in its proper position. Note that if two corner cubelets are properly positioned and share a face, we are in case (i.).

a. Properly orient cube.

1. if the properly positioned corner cubelet is in the  $ful$  position, Reorient: (F,L,B,R).
2. if the properly positioned corner cubelet is in the  $fur$  position, Reorient: (R,L)(F,B).
3. if the properly positioned corner cubelet is in the  $bur$  position, Reorient: (F,R,B,L).
4. if the properly positioned corner cubelet is in the  $bul$  position, then continue to next step.

- b. Then one or two applications of  $[FR] U [RF] U^{-1} (R^{-1}F(U^2R^2)^3F^{-1}R)$  should properly position the other three corner cubelets.
    - ii. It is possible to properly position only two corner cubelets on a diagonal.
      - a. Properly orient the cube.
        - 1. If the properly positioned corner cubelets are in the *ful* and *bur* positions, then continue to (b.).
        - 2. If not, Reorient: (F,L,B,R).
      - b. Next do  $[FR] U [RF] U^{-1} (R^{-1}F(U^2R^2)^3F^{-1}R)$ . Now we are in case (i.) from above. So complete case (i.) to properly position the corner cubelets.
    - iii. All corner cubelets are in their proper position! Carry on.
- III. Orient remaining corner cubelets. Reorient: (U,F,D,B). Note that after reorientation the current Front face was originally the Down face.
- A. Either *none*, *1*, *2*, or *all* will be properly oriented.
- i. If none of the Front corner cubelets are properly oriented:
    - a. Find the facelets that should be Front on the *ful<sub>o</sub>* and *fdl<sub>o</sub>* cubelets.
      - 1. If both or neither are facing left, then one or two applications of  $[FR]^2 R^{-1} [RD]^2 R$  will properly orient the *ful<sub>o</sub>* and *fdl<sub>o</sub>* cubelets.
      - 2. If not Reorient: (L,U,R,D); then one or two applications of  $[FR]^2 R^{-1} [RD]^2 R$  will properly orient the *ful<sub>o</sub>* and *fdl<sub>o</sub>* cubelets.
    - b. Next Reorient: (R,L) (U,D).
    - c. Repeat (a.) then all corner cubelets should be properly oriented in their proper positions.
  - ii. If one front corner cubelet is properly oriented.
    - a. The properly oriented cubelet is in one of the following positions: *dfr*, *dfl*, *ful*, *fur*.
      - 1. If in the *dfr* position, Reorient: (R,U,L,D).
      - 2. If in the *dfl* position, Reorient: (R,L) (U,D).
      - 3. If in the *ful* position, Reorient: (R,D,L,U).
      - 4. If in the *fru* position continue to next step.
    - b. Then one or two applications of  $[FR]^2 R^{-1} [RD]^2 R$  will properly orient the *ful<sub>o</sub>* cubelet.
    - c. Reorient: (U,R,D,L).
    - d. Then one or two applications of  $[FR]^2 R^{-1} [RD]^2 R$  will properly orient the *ful<sub>o</sub>* and *fdl<sub>o</sub>* cubelets.
  - iii. If two Front corner cubelets are properly oriented. They either share a side or are on the diagonal.
    - a. If they share a side:
      - 1. Fix the front face and reorient so that the shared side is the right side.

2. One or two applications of  $[\text{FR}]^2 \text{R}^{-1} [\text{RD}]^2 \text{R}$  will properly orient the  $\text{ful}_o$  and  $\text{fdl}_o$  cubelets.
- b. If the two properly oriented cubelets are on a diagonal:
1. Fix the front face and reorient so that the properly oriented cubelets are in the  $\text{fur}$  and  $\text{dfl}$  positions.
  2. Then one or two applications of  $[\text{FR}]^2 \text{R}^{-1} [\text{RD}]^2 \text{R}$  will properly orient the  $\text{ful}_o$  cubelet.
  3. Reorient: (U,R,D,L).
  4. Then one or two applications of  $[\text{FR}]^2 \text{R}^{-1} [\text{RD}]^2 \text{R}$  will properly orient the  $\text{ful}_o$  and  $\text{fdl}_o$  cubelets.

IV. Position Middle edge cubelets. Reorient: (F,D,B,U). Note that after reorientation the Up and Down faces are the original Up and Down faces.

A. Find  $\text{fr}_o$ .

- i. If in Middle layer,  $\text{fr}_o$  is in one of the following positions:  $\text{br}$ ,  $\text{bl}$ ,  $\text{fl}$ ,  $\text{fr}$ .
  - a. If  $\text{fr}_o$  is in  $\text{br}$ , use  $(\text{R}^2\text{D}^2)^3$  to properly position  $\text{fr}_o$ .
  - b. If  $\text{fr}_o$  is in  $\text{bl}$ , use  $(\text{R}^2\text{D}^2)^3 (\text{B}^2\text{D}^2)^3 (\text{R}^2\text{D}^2)^3$  to properly position  $\text{fr}_o$ .
  - c. If  $\text{fr}_o$  is in  $\text{fl}$ , use  $(\text{F}^2\text{D}^2)^3$  to properly position  $\text{fr}_o$ .
  - d. If  $\text{fr}_o$  is in  $\text{fr}$ , continue to (iii.).
- ii. If  $\text{fr}_o$  is in Down layer, it is in one of the following positions:  $\text{df}$ ,  $\text{dr}$ ,  $\text{db}$ ,  $\text{dl}$ .
  - a. If  $\text{fr}_o$  is in  $\text{df}$ , use  $\text{D}^{-1}\text{L}^{-1}\text{D}^{-1}(\text{F}^2\text{D}^2)^3 \text{DLD}$  to properly position  $\text{fr}_o$ .
  - b. If  $\text{fr}_o$  is in  $\text{dr}$ , use  $\text{D}^2\text{L}^{-1}\text{D}^{-1}(\text{F}^2\text{D}^2)^3 \text{DLD}^2$  to properly position  $\text{fr}_o$ .
  - c. If  $\text{fr}_o$  is in  $\text{db}$ , use  $\text{DL}^{-1}\text{D}^{-1}(\text{F}^2\text{D}^2)^3 \text{DLD}^{-1}$  to properly position  $\text{fr}_o$ .
  - d. If  $\text{fr}_o$  is in  $\text{dl}$ , use  $\text{L}^{-1}\text{D}^{-1}(\text{F}^2\text{D}^2)^3 \text{DL}$  to properly position  $\text{fr}_o$ .
- iii. Reorient: (F,L,B,R). Repeat (A.) until all of the Middle edge cubelets are properly positioned.

V. Orient Middle edge cubelets.

A. Either none, one, two, three or all four Middle edge cubelets are properly oriented.

- i. If none are properly oriented, Reorient: (U,L,D,R).
  - a. Then do  $(\text{F}^2\text{R}^2)^3 (\text{RD}^{-1}\text{RFL}^{-1}\text{F}) (\text{F}^2\text{R}^2)^3 (\text{F}^{-1}\text{LF}^{-1}\text{R}^{-1}\text{DR}^{-1})$ . Reorient: (F,B) (U,D).
  - b. Then do  $(\text{F}^2\text{R}^2)^3 (\text{RD}^{-1}\text{RFL}^{-1}\text{F}) (\text{F}^2\text{R}^2)^3 (\text{F}^{-1}\text{LF}^{-1}\text{R}^{-1}\text{DR}^{-1})$  to properly orient the remaining middle edge cubelets.
  - c. Reorient: (U,L,D,R).
- ii. If one of the middle edge cubelets is properly oriented Reorient the cube. Fix the Up face and the reorient so that the properly oriented middle edge cubelet is in the  $\text{bl}$  position.
  - a. Reorient: (U,L,D,R).
  - b. Do  $(\text{F}^2\text{R}^2)^3 (\text{RD}^{-1}\text{RFL}^{-1}\text{F}) (\text{F}^2\text{R}^2)^3 (\text{F}^{-1}\text{LF}^{-1}\text{R}^{-1}\text{DR}^{-1})$ .
  - c. Reorient: (U,L,B) (F,D,R).
  - d. Do  $\text{U}^{-1} [(\text{F}^2\text{R}^2)^3 (\text{RD}^{-1}\text{RFL}^{-1}\text{F}) (\text{F}^2\text{R}^2)^3 (\text{F}^{-1}\text{LF}^{-1}\text{R}^{-1}\text{DR}^{-1})] \text{U}$ .

- e. Reorient (F,U,B,D)
- iii. If two of the middle edge cubelets are properly oriented, either the two oriented cubelets share a face or they do not.
  - a. If they share a common face Reorient by fixing the Up face, and send the common face to the Back.
    - 1. Reorient: (U,L,D,R).
    - 2. Do  $(F^2R^2)^3 (RD^{-1}RFL^{-1}F) (F^2R^2)^3 (F^{-1}LF^{-1}R^{-1}DR^{-1})$  to properly orient the other middle edge cubelets.
    - 3. Reorient: (U,L,D,R)
  - b. If they do not share a common face, fix the Up face and reorient so that one of the properly oriented middle edge cubelets is in the *fr* position.
    - 1. Reorient: (U,L,D,R).
    - 2. Do  $(F^2R^2)^3 (RD^{-1}RFL^{-1}F) (F^2R^2)^3 (F^{-1}LF^{-1}R^{-1}DR^{-1})$ .
    - 3. Reorient: (U,F,D,B).
    - 4. Then do  $(F^2R^2)^3 (RD^{-1}RFL^{-1}F) (F^2R^2)^3 (F^{-1}LF^{-1}R^{-1}DR^{-1})$  again, and now all of the middle edge cubelets should be properly oriented.
    - 5. Reorient: (U,L,D,R)
- iv. If three middle edge cubelets are properly oriented, then fix the Up face and reorient so that the non-oriented middle edge cubelet is in the *fl* position.
  - a. Reorient: (U,B,D,F).
  - b. Then do  $U^{-1}[(F^2R^2)^3 (RD^{-1}RFL^{-1}F) (F^2R^2)^3 (F^{-1}LF^{-1}R^{-1}DR^{-1})]U$  to properly orient the last middle edge cubelet.
  - c. Reorient: (U,B,D,F).
- v. If all four are properly oriented Reorient: (U,D) (R,L).

#### VI. Position Up (originally Down) edge cubelets.

- A. Either none, one or all will be in proper position.
  - i. If none are in proper position, do  $(U^2R^2)^3 (B^{-1}UB) (U^2R^2)^3 (B^{-1}U^{-1}B)$ , then one of the edge cubelets will be properly positioned. Proceed to (ii.).
  - ii. If one is properly positioned, fix the Up face and reorient so that the properly positioned cubelet is in the *ul* position.
    - a. Then one or two applications of  $(U^2R^2)^3 (B^{-1}UB) (U^2R^2)^3 (B^{-1}U^{-1}B)$  will properly position all of the edge cubelets.
  - iii. If all are properly positioned carry on.

#### VII. Orient Up (originally Down) edge cubelets. Reorient: (U,F,D,B).

- A. Either none, two or all of the edge cubelets will be properly oriented.
  - i. If none:
    - a. Do  $(F^2R^2)^3 (RD^{-1}RFL^{-1}F) (F^2R^2)^3 (F^{-1}LF^{-1}R^{-1}DR^{-1})$ .
    - b. Then do  $F [(F^2R^2)^3 (RD^{-1}RFL^{-1}F) (F^2R^2)^3 (F^{-1}LF^{-1}R^{-1}DR^{-1})] F^{-1}$  (then the Rubik's Cube should be solved!).

- ii. If two of the edge cubelets are properly oriented, then the two oriented cubelets will either be opposite each other or adjacent to each other.
  - a. If the two properly oriented edge cubelets are opposite each other, fix the Front face and reorient so that the non-oriented cubelets are in the *fu* and *fd* positions.
    1. Do  $(F^2R^2)^3 (RD^{-1}RFL^{-1}F) (F^2R^2)^3 (F^{-1}LF^{-1}R^{-1}DR^{-1})$  (then the Rubik's Cube should be solved!).
  - b. If the two properly oriented edge cubelets are adjacent to each other, fix the Front face and reorient so that they non-oriented cubelets are in the *fu* and *fr* positions.
    1. Do  $R^{-1}D^{-1} [(F^2R^2)^3 (RD^{-1}RFL^{-1}F) (F^2R^2)^3 (F^{-1}LF^{-1}R^{-1}DR^{-1})] DR$  (then the Rubik's Cube should be solved!).
- iii. If all of the edge cubelets are properly oriented then proceed to (VIII).

VIII. Now celebrate and be pleased with yourself! On the off chance it didn't work, try again after a little break.