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Problem Set #2

3). Develop all of the processes necessary to solve the TopSpin game and write up an algorithm for solving it. Include a justification for how many of the states are reachable and, if possible, which permutations in S_{20} they represent.

Algorithm for Solving TopSpin:

Note: Orienting the game with the rotating purple disk on top, we designate the four positions located on the disk as, from left to right, Position numbers 1, 2, 3, and 4. The notation used is as follows: F is a 180 degree rotation of the purple disk, thus having the cycle notation $F = (1,4)(2,3)$; R is a clockwise rotation of all 20 numbers by one position, so that the number at Position #1 will now be at Position #2, and so on. Thus, R^{-1} is a counter-clockwise rotation by one position. The statement "$n-1$ is exactly four positions away from n" means that the process R^4 would place $n-1$ in the position previously occupied by n. If $n-1$ is exactly three positions away from n, R^3 would place $n-1$ where n used to be, and so on.

1. First, we order the numbers 5 through 20.
   (a) Let n be the smallest number such that all numbers n, $n+1$, . . ., 20 are in the correct consecutive order from smallest to largest clockwise. Find the next consecutive number $n-1$ and perform the necessary number of rotations, R, to place $n-1$ at Position #1.
      i. If $n-1$ is more than four positions away from n, perform the process FR^{-3}. Repeat or continue on as necessary.
      ii. If $n-1$ is exactly four positions away from n, perform the process F. The number $n-1$ should now be in consecutive order with n. Return to part (a) as necessary.
      iii. If $n-1$ is exactly three positions away from n, perform the process $R^2FR^{-1}F$. The number $n-1$ should now be in the correct position next to n. Return to part (a) as necessary.

iv. If $n-1$ is exactly two positions away from n, perform the process $R^2FRFR^{-1}F$. The number $n-1$ should now be in the correct position next to n. Return to part (a) as necessary.

2. We now order the remaining four numbers, 1 through 4.

   (a) Note: the general process to switch the two numbers in Positions #2 and #3, to which we will refer, is $R^3(FR)^{17}R^{-2}FRFR^{-1}FR^{-1}$.

   (b) Perform as many rotations, R, as necessary to place the numbers 1, 2, 3, and 4 in Positions #1, 2, 3, and 4, not necessarily respectively. Note the Position number of 4. If 4 is in Position #1 or #2, perform the action F to place 4 in Position #3 or #4. If 4 is in Position #4, continue on to Part (d).

   (c) If the number 4 is in Position #3, perform the action R^{-1} to place 4 in Position #2. Then refer to Part (a) to switch the number 4 with the number in Position #4, and perform the rotation R to return the number 4 to Position #4.

   (d) If the number 3 is in Position #2, refer to Part (a) to place the number 3 in Position #3 and continue to Part (e) if necessary. If the number 3 is in Position #1, perform the action R to place the number 3 in Position #2. Then perform the process in Part (a) and then the action R^{-1} to place the number 3 in Position #2. Refer again to Part (a) to place the number 3 in its correct order in Position #3. Continue on to Part (e) if necessary.

   (e) If the number 2 is in Position #1, perform the action R, complete the process in Part (a), and then perform the action R^{-1} to return the number 2 to its correct order in Position #2.

At this point, all twenty numbers should be in the correct, consecutive order. From this algorithm, we can deduce that all 20! states are reachable. This follows from the process that allows two adjacent numbers to be switched. By repeating this process with the appropriate pairs of numbers, it is possible to place any number in any desired position. Thus, all possible states can be obtained.