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Math 434

Problem 2.3
Develop all of the processes necessary to solve the TopSpin game and write up an algorithm for solving it. Include a justification for how many of the states are reachable and, if possible, which permutations in S_{20} they represent.

Holding the topspin game with the purple disc away from you, let the counter-clockwise most position in the purple disc be the 1 position (left most position with disc away from you). Let the position clockwise from the 1 position be the 2 position, and so on for all twenty positions. Note the 20 position is one position counter-clockwise from the 1 position.

Let g and h be elements of S_{20}.
Let g = (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20)
Let h = (1,4) (2,3)

Note that g is equivalent to moving all of the pieces one position in a clockwise direction, and h is equivalent to a 180-degree rotation of the purple disc. So g and h generate the TopSpin group G, and G is a subgroup of S_{20}.

Note:

$$g^{-1} = (20,19,18,17,16,15,14,13,12,11,10,9,8,7,6,5,4,3,2,1)$$
$$h = h^{-1}$$
$$|g| = 20 \text{ and } |h| = 2$$

As with the Rubik's Cube, let $[hg] = hgh^{-1}g^{-1}$. Therefore $[hg^{-1}] = hg^{-1}h^{-1}g$.

$$[hg^{-1}] = (1,3,5,2,4)$$
$$\text{So } [hg^{-1}]^2 = (1,5,4,3,2)$$
$$([hg^{-1}]^2 g^{-4})^5 = (2,20,19,18,17,16,15,14,13,12,11,10,9,8,7,6,5,4,3)$$
$$\text{Then } ([hg^{-1}]^2 g^{-4})^5 g = (1,2)$$

With this as an inspiration, we can permute one with anything in G.

$$\text{let } a = ([hg^{-1}]^2 g^{-4})^5.$$
$$\text{Then } (1,2) = ag$$

$$\text{We claim } (1,n) = a^{n-1}(g^2a)^{n-2}g \qquad \text{for } 1 < n \leq 20$$

Note that a^2 = (20,18,16,14,12,10,8,6,4,2,19,17,15,13,11,9,7,5,3), i.e. a^2 fixes 1 and shifts everything else two positions counter-clockwise (except 2 and 3 which move three positions counter-clockwise in order to skip 1). Essentially we have put 1 between 3 and 4. Similarly a^{n-1} will fix 1 and move everything else n-1 positions counter-clockwise putting 1 between the n and n+1.

Now in order to switch 1 and n we next need to put n between 20 and 2. Recall 1 is still in the 1 position and n is in the 20 position.

Note $g^2a = (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19)$, i.e. g^2a fixes the twenty position and moves everything else one position clockwise. So g^2a will put n between n-2 and n-1, $(g^2a)^2$ will put n between n-3 and n-2, and $(g^2a)^m$ will put n between n-(m+1) and n-m. In particular $(g^2a)^{n-2}$ will put n between 1 and 2. But recall that we have already put one between n and n+1, so 20 is next to 2. Thus after $a^{n-1}(g^2a)^{n-2}$, n is between 20 and 2 and since we moved n away, 1 is between n-1 and n+1. Finally n is in the 20 position and we want it in the 1 position, so g simply moves everything one position clockwise.

Furthermore we can permute any two distinct positions in G.

For any m, n with $1 \leq m < 20$, $1 < n \leq 20$, and $m < n$.

$$(m,n) = g^{1-m}\,(a^{(n-m)}\,(g^2a)^{(n-m-1)}\,g)\,g^{m-1}$$

Note (m,n) = (n,m) for any m and n.

Now we can show that $G = S_{20}$. Recall that G is a subgroup of S_{20}. Thus if we can show that S_{20} is a subset of G we are done. Let k be an arbitrary element in S_{20}. Then k is a permutation and can be written in cycle notation. Each disjoint cycle of k may be written $(n_1, n_2, \dots, n_i)$ where $1 < i \leq 20$. The cycle $(n_1, n_2, \dots, n_i)$ can be rewritten as a product of transpositions, specifically $(n_1, n_i)\,(n_1, n_{i-1}) \dots (n_1, n_3)\,(n_1, n_2)$. Since we can permute any two distinct positions in G, we can reach k. Thus $G = S_{20}$; that is every permutation in S_{20} can be reached on the TopSpin game.

An algorithm to solve the TopSpin game.
Let 1_o denote the piece with 1 written on it and 2_o denote the piece with two written on it and so on for all twenty pieces.

I. Positioning the first sixteen pieces.
   A. Find 2_o and using an element of <g> move it to the 4 position (the clockwise most position in the purple disc).
      i. If 1_o is **not** in the 1, 2, or 3 position then do hg^3, and repeat i. until 1_o is in one of the 1, 2, or 3 positions.
      ii. If 1_o is in one of the 1, 2, or 3 position:
         a. If 1_o is in the 1 position do $(g^{-2}h)\,(gh)$ to position 2_o next to 1_o.
         b. If 1_o is in the 2 position do $(g^{-2}h)\,(g^{-1}h)\,(gh)$ to position 2_o next to 1_o.
         c. If 1_o is in the 3 position then 2_o is next to 1_o, so continue to B.

   B. Repeat A replacing 2_o for 1_o and 3_o for 2_o. Continue repeating A in this manner, positioning the next highest piece, until 16_o is properly positioned next to 15_o.

II. Positioning the remaining four pieces (recall $a = ([hg^{-1}]^2\,g^{-4})^5$).
   A. Using an element of <g> (probably g^{-1}) place 16_o in the 20 position.

i. If 17_o is in the 1 position proceed to B.
ii. If 17_o is in the 2 position do ag to position 17_o next to 16_o.
iii. If 17_o is in the 3 position do $g^{-1}ag^{2}ag$ to position 17_o next to 16_o.
iv. If 17_o is in the 4 position do h to position 17_o next to 16_o.

B. Using an element of <g> (probably g^{-1}) place 17_o in the 20 position.
   i. If 18_o is in the 1 position proceed to C.
   ii. If 18_o is in the 2 position do ag to position 18_o next to 17_o.
   iii. If 18_o is in the 3 position do $g^{-1}ag^{2}ag$ to position 18_o next to 17_o.

C. Using an element of <g> (probably g^{-1}) place 18_o in the 20 position.
   i. If 19_o is in the 1 position then do g^{-2} to finish.
   ii. If 19_o is in the 2 position do ag^{-1} to finish.