Noah Chang
Math 434
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Problem 2.3

Problem:

Develop all of the processes necessary to solve the TopSpin game and write up an algorithm for solving it. Include a justification for how many of the states are reachable and, if possible, which permutations in S_{20} they represent.

Solution:

First we define some notation. We define a solution to the TopSpin to be when the numbered discs are in consecutive order, from left to right, with 1 in the farthest left of the four positions in the turnstile, when the turnstile is at the top of the track. Thus the discs which occupy the four positions in the turnstile are, from left to right, 1, 2, 3, and 4, with the numbers continuing in order clockwise around the track, ending with 20 on the left side of 1. We label the positions in the turnstile, from left to right, A, B, C, and D. Thus, when the puzzle is solved, 1 is in the A position, with 2, 3, and 4 in the B, C, and D positions respectively. We will call these positions the "home row."

The two basic moves in the puzzle are turning the turnstile, or shifting the numbers around the track. We will call one turn of the turnstile T. It is clear that, whether you turn the turnstile left or right, the effect is the same on the home row. A is swapped with D and B is swapped with C. Thus, in cycle notation, the effect on the puzzle from a solved state is

$$T = (1,4)(2,3)$$

with the 16 other discs remaining fixed. We see that $T^2 = I$ and that $T^{-1} = T$. We will call a shift to the right (clockwise) R. It is clear that every disc remains in consecutive order when R is performed, simply shifted. Thus, from a solved state,

$$R = (1,2,3,...,19,20)$$

In the same way, we call a shift to the left L. Note that $L = R^{-1}$ and that

$$L = (1,20,19,...3,2)$$

It is clear that $R^{20} = L^{20} = I$.

These are the three basic permutations that will be used to solve the puzzle. The basic algorithm is as follows: Starting with disc 1, get the first sixteen discs (1-16) in consecutive order. Then the remaining four discs (17-20) will be in any of $4! = 24$ possible permutations. We will use a permutation that swaps any two adjacent discs to get these last four discs in order.

We begin by getting the first sixteen discs in order. We begin by placing disc 2 adjacent and to the right of 1. To do this, regardless of the position of any other discs, place 2 in the A position of the home row (the farthest left). We then shift disc 2 to the D position and perform a turn (R^3T). We will call this permutation α. This is simply reducing the number of discs between 1 and 2. Continue to perform α until 2 approaches closely to 1. Each time T is performed, 2 becomes three discs closer to 1. We are not concerned with the other discs yet, so we will not consider how they are affected under α.

Once 2 is three or more discs closer to 1 (i.e. there are three or less discs between 2 and 1), we must choose the correct permutation to place 2 adjacent to 1. The easiest possibility is if 2 is exactly three discs away from 1. Then simply performing α will place 2 adjacent to 1. If this is not the case, we will need another permutation defined as $\delta = RTL^2$. This permutation shifts 2 into the B position and then places it in the C position by turning the turnstile, and finally shifts 2 back into the A position of the home row. The purpose of this permutation is to increase the number of discs between 1 and 2 by one. Thus if there are two discs between 1 and 2, then performing $\delta\alpha$ will place 2 adjacent to 1 by increasing the number of discs between them from two to three, creating the easiest case and allowing α to finish the job. If there is only one disc between 1 and 2, δ will have to be performed twice before α can be used ($\delta^2\alpha$).

This is the algorithm that will be used to place discs 1-16 in order. It should be clear that disc 1 is fixed by δ. Thus, since δ only affects disc 2 and those which are larger, we are not concerned with how the other discs are affected under δ.

To summarize: Place the current disc (in the previous example, disc 2) in the A position of the home row, and perform α until it is three or less discs away from the target disc, which is the smaller disc that it belongs adjacent to (in the previous example, disc 1). Let the number of discs between the current disc and the target disc be k. Perform $\delta^{3-k}\alpha$ in order to place the current disc adjacent to the target disc. Starting with disc 2, use this algorithm until the first sixteen discs are in consecutive, increasing order.

It should be clear that once a disc placed where it belongs, it will remain fixed, since α always shifts right. Thus the "solved" discs will always be moving counterclockwise, and will never enter the turnstile again. As we get close to wrapping around the track, back to the disc 1, we do not need to worry about messing up the part of the puzzle that we have already solved, since we are only concerned with the first sixteen discs. Thus, once they are in order, the last four discs (17-20) will be left, probably out of order, but discs 1-16 will always be in order, because the turnstile holds four discs. Thus disc 1 will never be affected when placing the first sixteen discs in order. We will now explore how to solve the last four discs.

It is clear that a permutation that swaps any two adjacent discs but fixes the other eighteen could be used to solve the puzzle from the beginning. It would be tedious, because you would have to move a disc around the track by continually swapping it with an adjacent disc until it reached its proper position. However, when dealing with the last four discs, this technique is necessary. We could continue in the same fashion as before and place the last four discs in order using α and δ, but since the discs wrap around the track back to disc 1, discs 1-4 would be affected when placing discs 17-20 in order this way. Since we do not want to mess up the first sixteen discs which are already in order, we use a more complicated permutation.

[Note: After figuring out the previous portion of the algorithm and performing it on various permutations of the puzzle, I came across a permutation with two of the last four discs in their proper position, but the remaining two were adjacent but backwards. Thus I knew there had to exist a permutation to swap two adjacent discs, otherwise the puzzle would not be solvable. After spending a couple of hours trying to figure the permutation out, I was unsuccessful. Finally I resorted to looking the permutation up (Bennett, 13).]

The permutation for swapping two adjacent discs is as follows: Place the two discs to be swapped in positions C and D of the home row. Then perform $TRTLTRTR^3$. We will call this permutation β. We explore this permutation by starting with the puzzle in a solved state. Thus discs 3 and 4 are in the C and D positions of the home row, respectively. By simply tracing β through we see that

$$\beta = (1, 6, 10, 14, 18, 2, 7, 11, 15, 19, 3, 8, 12, 16, 20, 5, 9, 13, 17)$$

Note that disc 4 is the only disc fixed by β. Tracing β through a few more times, we find that

$$\begin{aligned}
\beta^2 &= (1, 10, 18, 7, 15, 3, 12, 20, 9, 17, 6, 14, 2, 11, 19, 8, 16, 5, 13) \\
\beta^3 &= (1, 14, 7, 19, 12, 5, 17, 10, 2, 15, 8, 20, 13, 6, 18, 11, 3, 16, 9) \\
\beta^4 &= (1, 18, 15, 12, 9, 6, 2, 19, 16, 13, 10, 7, 3, 20, 17, 14, 11, 8, 5) \\
\beta^5 &= (1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20)
\end{aligned}$$

By performing L after β^5, we see that every element is fixed except for discs 3 and 4. Specifically, we see that $\beta^5 L = (3, 4)$.

Using $\beta^5 L$, we can solve the last four discs without affecting the first sixteen. It is best to continue the overall algorithm of getting the discs in their proper position, in order. Thus starting with disc 17, $\beta^5 L$ would have to be performed, at most, 4 times.

To summarize: In order to solve the last four discs, place the two discs that need to be swapped in positions C and D of the home row, and perform $\beta^5 L$. Start with disc 17 and continue swapping discs to place them in consecutive, increasing order.

It is clear that the ability to swap two adjacent discs automatically allows us to swap any two discs in the puzzle. We would simply swap the first disc with the ones adjacent to it until it was next to the second disc, then swap those two discs, and finally swap the second disc backwards in the direction from which the first disc came, until it reached the original position of the first disc. Thus, viewing the set of twenty discs as S_{20}, we see that any permutation of S_{20} can be represented by a state of the puzzle. Since $\beta^5 L$ is an odd permutation, we see that every possible state is indeed reachable.

References:

Bennett, Curtis D. "TopSpin on the Symmetric Group." *Math Horizons* Sept. 2000: 11-15.