La La Logic

Music Theory as an Axiomatic System and Aesthetic Logic

By

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"Music is emotions in their logical order" ~Anonymous

Overture

Is music logical? Can an art form that relies on aesthetic interpretation of its value be said to have a set of overriding principals that govern its functions and creation? The answer is both yes and no. Western music is governed by rules and theory that define its production and logical quality. This applies especially to music of the Renaissance, Baroque and Classical Eras. It was during the Baroque period that music theory was first solidified and “axiomatized”. Classical music theory has laws and theorems as stringent as those of geometry. The development of musical logic in the last 400 years though has followed a path similar to that of the development of non-Euclidean geometry. Just as geometry explored new worlds and planes, music theory became increasingly unimportant. In the late nineteenth and early twentieth century it was as if western music had developed into a new system; it was similar to the old way in its basic elements, but seemed to exist on a different plane. Classic and Baroque music can be studied as a form that is almost a model of an axiomatic system in itself, and a geometric system at that. But music, like geometry, evolved. As music evolved, so did its logic; a logic that was not just axiomatic, but aesthetic.

First movement:

MUSIC AS AN AXIOMATIC SYSTEM

Undefined terms:

To prove that music and geometry are comparable as axiomatic systems, let us try to define music in terms of the axioms and vocabulary of geometry, this will included the undefined terms and the existence of logical implications from the axioms we form. An axiomatic system must contain at least one undefined term. Our first undefined term is more on an aesthetic level, but none the less is something that can't be defined totally: **Music** itself. What is music, really? Music can be different things for different people; therefore no one can truly define it. Its definition is in the perceptions of human beings themselves. The next term is the very basis of music and equally as hard to define: a **note**. We can define a note as a pitched sound, but that would include all sounds within the range of human hearing in the definition. That same spectrum of human hearing itself cannot process all sounds and pitches, therefore our musical plane of geometry is automatically finite, limited by what we can hear. A note itself is a preset pitch that does not change or fluctuate. A certain note is defined by the frequency of vibration and that does not change. Though composers of the twentieth century used every sound from car horns to coughing as "music", this does not suit our purposes. Therefore, it's best to have a note, like a point, as an undefined term. In this system, a note will be analogous to a point. If we are going to look at music as not just an axiomatic system, and *western* music as a model of that system, we should better define our "points". Within the world there are different systems of notes and tones (a tone is a term for the sound of a note), just as there are different models of, say, incidence geometry. In Asia a series of 5 notes, a "pentatonic" scale, is used. In Indian and Arabic music, they use a system of "quarter tones" that are divided by much smaller distances of pitch than typical western music. These different types of music can be viewed as separate models of music, or separate planes (like elliptic versus Euclidean). The plane we will use is the western one that was standardized in the renaissance and used for the music theory of that era and the baroque, the most structured of western musical eras. We will call this the **Western model**, because it is the sound familiar to western culture. The system of logic we will invoke we shall call the **Baroque system**, since that was the era in which it had the most power and was most strictly obeyed. It should be restated that this is a *finite* plane, so every line and such has limits. A finite plane made of distinctly separate points (and notes are distinctly separate) also eliminates the need for a continuity axiom. This model uses a set of twelve different notes. These notes can exist in several different **octaves** as well. Octaves are sets of the same notes but they sound different because they are either **higher** or **lower** in sound. Higher and lower are qualifiers that were invented by humans, and cannot be defined to someone who had never heard music, therefore we can make these two terms undefined. We can interpret higher to mean "vibrating faster" and lower as "Vibrating slower", and also call on the readers own sense of the terms. It might be said that octaves are the opposite of parallel lines, because the all have the same collection of points, or notes, but they are pitched higher or lower so they aren't the *exact* same notes. So, now as a dot is a point in Euclidean geometry we have chosen a pitched note anywhere in the twelve tone scale to be a point in our axiomatic system of music. Since there are just twelve notes, we can name them all now, but one side note (no pun intended) before we do. Depending on what **key** a piece is in, a note called one name in one piece might be called another name in another piece, but sound the same. This is because many notes have two names. These notes are called **enharmonic equivalents** (D flat is the same note as C sharp). Though this may seem like jumping ahead, we can take this moment to discuss congruence. We can leave congruence undefined, but in this model interpret it as being enharmonically equivalent, or more simply: sounding the same. So let us list or notes starting on C (the traditional "neutral" key and starting point), this is the order (Alphabetical) that these notes are always arranged it and *cannot* be changed, C, C sharp/D flat, D, D sharp/E flat, E, F, F sharp/G flat, G, G sharp/A flat, A, A sharp/B flat, B. If a note is in the same octave as another and is latter in this order than the other note it is said to be higher in pitch, if is comes before in the order it is said to be lower in pitch. If one note is in a higher or lower octave than another, the question of whether it is higher or lower is easily answered. And there we have the complete twelve tone scale. But in the baroque system, most music was not written using all twelve notes; composers would use sub-sets of this large group to create keys. A **Key** is a subset of seven notes from the big twelve, organized in specific relationships. These relations ships were defined by the distances between the seven notes. Each note in the twelve tone scale is a **half step** apart, in a key the distances are half steps and also **whole steps**, the distance between two notes in the 12 tone scale. One qualification for a key you can have only one kind of each note: i.e. if you have a key with E in it, you can't have E sharp or E flat. The notes stay in the same order as above with one of each letter. The points in the key of C major are C, D, E, F, G, A, B. This key is also defined by the relationships between these notes, but more on that later down the line. Speaking of lines:

In geometry – Euclidean or non-Euclidean – the next undefined term is a line. If a note is equivalent to a point, does that mean that a line also has a musical equivalent? In music a line is more easily defined than in geometry. In geometry a line cannot be defined, even in terms of its points because the numbers of points on that line is infinite; but not so in music. Because of continuity, we know that there is no place on a geometric line without points on it; essentially, there aren't any holes in the line. This is not the case in music; no matter how we define a line there will be holes in it because not every possible pitch will be sounded in that line (if we were working with Indian or Arabic music, this might be possible, but we aren't, so it's just not going to happen). In music there are many things we could consider as lines. The easy solution would be that a musical line is simply a melodic line. A **melody** is simply a series of notes sounded separately. This would be the easy way out, and would allow parallel lines to exist through parallel melodic lines. But, what about non-collinear points? There would be no points not on the line, and therefore a very important element of geometry would be missing. So what else could be a line be? I believe it could be a **chord** (not the circle kind). A chord is a series of notes played together to create a new sound from the combination of those notes. Notes played together like this are said to be **in harmony**, and harmony is the basis for western music. It is for this reason that we will have a line be defined as a chord. This makes it so a line does not need to be an undefined term in musical axioms, because it is defined in terms of notes. Since there are limits to this line and the points it can contain, the obstacle of infinite lengths and infinite numbers of points on a line is therefore solved in the finite musical plane.

Incidence in Music

Though composers wrote great amounts of “incidental music” for plays, court functions and other reasons, we won’t be talking about those when we discuss how incidence will be discussed and defined in music. Incidence is an undefined term in geometry, but since we are using definitions in terms of notes and how they are written and sounded, we can overcome this difficulty.

Incidence geometry has much to do with whether a point lies on or off a line, since we have lines defined we can use that to explain incidence. A note might be incident with a chord if it is a member of, or sounded in that chord. But an even simpler way to say this is: two notes are incident if they are played in harmony. Since the set of points in music is finite, the relationships between them are finite as well, there are only so many distances between notes. In music these distances are called **intervals**. An interval can be one of two types harmonic or melodic. A **harmonic interval** consists of notes played together and a **melodic** one of notes played apart. Harmonic intervals are made up of incident points. This brings up the question, why aren’t all intervals chords or lines? We can solve this by saying that intervals are *segments*. Just as a line contains an infinite number of segments a chord contains many different intervals. Intervals are classified by how many **diatonic** (within the key) notes are between the two notes, counting those notes. The interval between C and D (count the notes, C and D, and not C shard/D flat, since that note can’t be in the key) is then called a second, not a zero. Intervals can also be perfect, major, minor, diminished or augmented. At the moment, only major, minor and perfect intervals matter, because augmented and diminished intervals are simply modifications of major, minor and perfect intervals, and all have major, minor and perfect enharmonic equivalents. Whether an interval is major or minor or perfect is specified by the number of half steps in-between the notes, when counting half steps we count *all* the possible notes, not just the diatonic ones. Remember how keys are defined by the distances of half and whole steps between the notes, those distance were intervals too. A half step is the same thing as a *minor* second, a whole step is two half steps put together, and this is called a major second, three half steps is a minor third and so on. Key relationships are made up of notes related by minor seconds and major seconds, or half steps and whole steps, as we said before. It is settled by definition that notes which are members of a chord will be said to be incident with that chord because they are part of the harmony established in that chord. Let us take one step further and say these notes are "**harmonically incident**". Now that we have a basic idea of what a key is and what harmony means, we can begin to examine how we construct chords, or lines, which is a process slightly more complex than saying two points define a line; in fact to make a complete, traditional chord, we need *three* points. Next we will define what a chord actually is and how the idea of "harmonic incidence" is paramount in the baroque system.

The creation of a chord is one of the foundations of our baroque system of music, and goes to the heart of music theory. How is a chord constructed? Is it made in the same fashion as a line in incidence geometry; two points have a line incident with them? Not really, it's slightly more complex than any two notes making a chord, or what is commonly accepted as a chord. In music of the early twentieth century composers did use this approach and smushed every conceivable combination of notes into the definition of a chord, leading to some very interesting musical sounds and creations. But since we are using traditional music theory as our axiomatic system, we will have a very precise definition of how a chord is constructed. As stated before, chords are made up of intervals (segments), one stacked on the other and so on. In the medieval era the main interval used in harmony was that of a fourth. The fourth is a **perfect interval** that has no real dissonant or harmonic edge to it (Fifths, eighths {octaves} and unisons {the same note played by two voices} are the only other perfect intervals), this quality in music leant itself to a very open and hallow sound, the kind characterized in Gregorian chant and other such music. But, as music evolved, the main interval of harmony became the third, an interval that allows a juicier and more complex sound. Traditional harmony is known as **tertiary** harmony; harmony made of thirds. Thirds are a much more harmonically inclined interval than the fourth. A chord is made of three notes separated by thirds. These notes must be chosen in alphabetical order, but once they are chosen they can be organized any way the composer pleases, these mix ups are called **inversions**. For example, to create a chord starting on A, we find the note a third above A – C, then the note a third above that – E. We now have a complete A chord. If we wanted to invert this chord we would have not have the A as the lowest pitch, if A was the lowest sounding pitch the chord would be said to be in **root position** . Chords, like intervals, can have different qualities, the same ones as intervals in fact – except for perfect. A major chord is made of a major third (four half steps) followed by a minor third (three half steps), a minor chord is the opposite: a minor third fallowed by a major one. Then there is a diminished chord, which is made of two minor thirds and an augmented chord which is made up of two major thirds - augmented chords are very uncommon in classical music. So, to make the chord we just constructed major, we need the interval between A and C to be major. If you count the half steps you will see it is not, so we must raise the C (Called the **third** of the chord) to C sharp. Then we must see how far away from the E (the **fifth** of the chord) is from C sharp. It is a minor third away. We now have an A major chord. This automatically gives us musical betweeness, for a note to be between two notes, one must be higher in pitch, the other lower, and this can be within a chord or a melody.

Non-incident notes and harmonic “triangles”

For a moment, let us think of our lines as sets (a popular model in its own right), and all notes not in the set of our chord as not incident with it. By this definition a note can be played at the same time as a chord and since it is not a member of that specific chord, still be non-incident. Such a practice exists in music; non-incident points that might seem, at first, to be part of the chord, but really aren’t. These are called **non-chord tones**. For example if we play the A major chord we created above and also play a B at the same time, that B is not a member of the chord because it obviously is not an A, a C sharp or an E. There are many types of non-chord tones that are defined in terms of the melodic lines to which they belong, but we need not go into their various distinctions. Their purpose is to create **dissonance** (i.e. not immediately pleasing or obviously harmonic), the purpose of which is to create tension and make the music more exciting and colorful.

There is one instance though, where a fourth note can be added to a chord. This note is found the same way as the other members of a chord, by going a third up from the highest pitch, so in our A major chord, the note a third up from E is a G (in the Key of A, G sharp exists instead but we are not actually using any keys at the moment). This type of chordal non-chord tone is called a **seventh**, and the chord it creates is called a **seventh chord**. A seventh is like a point not on a line that creates something new from that line and the point, since our lines are limited, we can consider seventh chords to be like triangles (not the percussion instrument, the geometric shape with nifty properties). Like a segment and a point not on the line containing that segment can create a triangle, so can a seventh added too a chord create a figure with it's own unique properties and rules. We can consider the three intervals in a seventh chord like angles, different combinations of them, like different combinations of angles in a triangle, create different sounds/shapes. Seventh chords have endless chapters devoted to their properties in various books, and properties that have no equivalents in geometry, as do normal chords. But these properties, none the less have their own logic. We have already defined congruence and do not need a continuity axiom. So what is left compare? We could spend ages explaining music theory and forcing geometric equivalents into it, but now we can explore how aspects in which the logic of geometry and music diverge, or at least seem to

Second Movement

AESTHETIC LOGIC

Musical logic involves which note goes where and why they do that; in that way music is different from geometry in that much of the logic is based of change over time. These rules were founded on the likes and dislikes of the people who invented them. It was a logic established on hearing something and saying "that sounds like it works". In geometry certain frustrations can arise by looking at a diagram and saying "what I want to prove is visible right there, why can't that just be accepted?". This sort or frustration arises from a certain aesthetic sense. In geometry the action of something must be proved precisely, we can't just say "that line wants to go though the other", but we can do that in music. A great many of the rules in music theory come from the simple principal of what sounds good. Certain types of chords usually lead to other types of chords because the sound of those chords creates a sort of tension that wants to be released in the resolution of another chord. Certain notes are said to "want" to go to other notes. This comes from the opinions of others that have been built into our minds, but the aesthetic sense is still there. Example, sing a simple scale on **solfege** (*do re mi* etc) ala "The Sound of Music", only stop at *ti*. Doesn't it sound incomplete, it's like the music *wants* to complete its cycle and go back to *do*? In our ears music leads us; the syllable *ti* is even called a **leading tone**. This is an aesthetic western view and preference that has been developed over the years. Music theory was built on aesthetic sense of what sounded right. Or to be more precise, what sounded good to a bunch of dead guys 300 years ago. But the principles still exist. There are actual scientific reasons for this, tied into the resonance of sound, but nothing of that sort was known when the rules we use today were set down. Breaking certain rules in music creates sounds that we would classify as strange or unappealing. Making a chord out of two notes that are only half step away from each other would be as musically illogical as forming a line out of one point. What we would call a contradiction in geometry we would call **dissonance** in music. It was mentioned before that dissonance is used in Baroque music, but it was used to create a tension that needed resolution. They used dissonance to get somewhere they wanted to go, like we would use a contradiction in a Reducto Ad Absurdum argument. But just as Euclidean geometry evolved into a system where seeming contradictions (multiple parallel lines, for example) could happily exist without excuses, so did music. This evolution came about in music in the late nineteenth and early twentieth century. Composers began to experiment and test the boundaries of what could be accomplished within traditional musical rules, and then they started making music that broke the rules, just as mathematicians broke rules and created new geometric worlds. Just as representations of non-Euclidean geometry seem illogical to our eyes and minds that have been raised on the idea of the world as Euclidean; modern music seemed aesthetically illogical to the first people that heard it. Igor Stravinsky was a great composer of the twentieth century, and his work grew more and more progressive as his career continued. He eventually wrote the sublimely dissonant and decidedly un-classical "Rite of Spring". This work was meant to be a representation of a primitive world and was originally a ballet. The reaction was much akin to the "howls from the Boetians" that met the invention of non-Euclidean geometry; there was literally a *riot* in the theater. Yet today, the Rite of Spring is viewed as a monumental work that is studied the world over, just as areas such as hyperbolic geometry are explored nowadays. Musical logic still owes much of its existence to popular aesthetics, but composition is by no means totally confined within the old rules anymore. Students are taught classical music theory because that's where it all began and what vast amount of classical western music is based upon. This is the same as the students who are taught Euclidean geometry all through middle and high school, only to be shocked to find that not everything fits into that mold. Music is unique in that it is fluid in its form; people have been bending and breaking the rules from the time the rules were created. When those rules were finally relegated to their own world, amazing and infinite musical horizons opened up to people. If we had stayed in the baroque era forever, we would have never heard the free and revolutionary sounds of Jazz or Leonard Bernstein.

Now, as said above, the science of geometry is not based on “it looks that way so it is that way”; a kind of *post hoc ergo proper hoc*, flawed logic. But is all geometry so very different from music in its aesthetic foundations? I believe that Euclidean geometry actually isn’t. In Euclidean geometry a triangle has 180 degrees, this might have been founded in the ancient’s perception that a triangle with more or fewer degrees didn’t *look* like a triangle should. In the middle ages the interval of the tritone (a very dissonant interval in between a perfect fourth and a perfect fifth and therefore cutting an octave in a perfect half) was banned by the church because it sounded like something that came out of hell. People who used tritones risked being burned at the stake for defying common aesthetics. It just sounded wrong. But now tritones are used everywhere and triangles that don’t have 180 degrees can exist. Rules in music evolved from how people heard and perceived what would sound good. We cannot know exactly what Euclid and his predecessors were thinking when they defined the geometry we used for so many thousand years, but it is possible that some of what they set down owed its existence to a simple aesthetic perception of how the world seemed to work. A line that met another at two points wouldn’t look like a line; it would defy the aesthetic logic that they had built up. As human aesthetics and perceptions evolved so did the limits of geometry and music. It is interesting to see how both non-Euclidean geometry and crazy modern music both began their true genesis in the eighteenth century. The perhaps one day we will look back on that era and our own and see them not as another renaissance but as a time when the idea of limitations and unbreakable ancient rules died. A time when we came to realize that in music, geometry, or any discipline the only limitations are in the mind, aesthetics and ideas of the artist.

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