# Drawing on Books

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**Abstract**

We discuss elementary graph theory, focusing on topological graph theory. We then discuss a type of topological surface known as a book. We define a graph, the *net*, based on Harary and Guy's *Möbius ladder*, and study its bookthickness. We then discuss two more methods of representing graphs embedded onto closed surfaces: rotation systems and band decompositions.

## 1 Elementary Graph Theory

A **graph** $G(V, E)$ is a set of **vertices** $V$ and a set of **edges** $E$ where each edge contains at most two vertices. There are several ways of representing a graph. We could represent it as a list of vertices and edges, or we could write the same information in a matrix, or we could draw vertices as dots and edges as curves between them. In this paper, we will represent graphs by lists or drawings. For example, Figure 1 shows the graph with vertices $\{a, b, c, d\}$ and edges$\{\{a, b\}, \{b, c\}, \{c, d\}, \{a, d\}, \{a, c\}\}$. The same graph can be drawn in multiple ways depending on how the drawer chooses to place dots and place and draw edges. In all realizations of a graph, however, the same vertices are always connected by the same edges.

A vertex is **adjoined** to an edge if it is one of the endpoints, and two vertices are **adjacent** if there is an edge containing both of them. The number of edges attached to a given vertex is the vertex's **degree**. In the graph of Figure 1, vertex $a$ has degree 3. If every vertex in a given graph has the same degree, then that graph is said to be **regular**. When an edge connects one vertex to itself, the edge is called a **loop**, and when two or more edges connect the same pair of vertices to each other, we say they are **multiple edges**. A graph with no loops or multiple edges is called **simple**. In general, properties belonging to simple graphs can be extended to graphs that are not simple, and so it is common to restrain one's interest (as we shall in this paper) to simple graphs only.

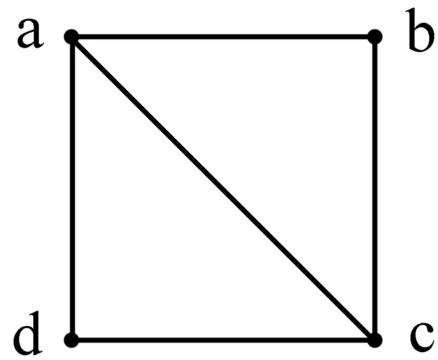


Figure 1: A graph

Some appealing families of graphs arise naturally. The most obvious is $K_n$, the **complete graph** on $n$ vertices. As its name suggests, this is the graph with $n$ vertices, each of which is connected to every other vertex by exactly one edge. $K_4$ and $K_5$ are pictured in Figure 2. Another interesting type of graph is a **bipartite** graph. This is a graph whose vertex set can be partitioned into two subsets such that every edge in the graph adjoins exactly one vertex from each set. In a **complete bipartite** graph, *every* vertex in one subset is connected to *every* vertex in the other. The bipartite graph in Figure 3 is $K_{2,3}$, the complete bipartite graph with two vertices in one subset and three in the other.

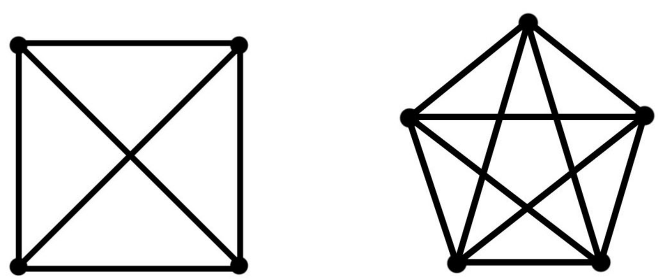

Figure 2: $K_4$, $K_5$

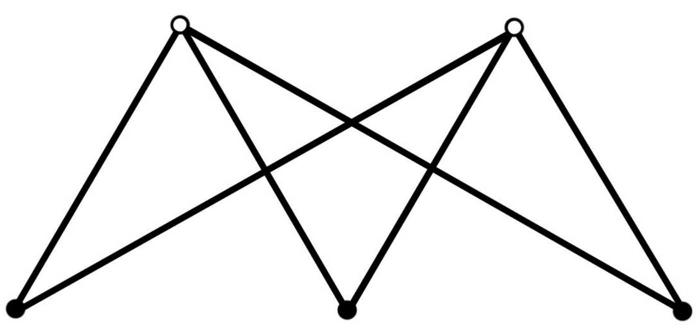

Figure 3: $K_{2,3}$

### 1.1 The Möbius ladder

There are many other common constructions of graphs, but we will only address one more at this point: the **Möbius ladder**, designed by Guy and Harary in 1967 [7]. The Möbius ladder $M_n$ is defined to be the graph on $n \geq 5$ vertices consisting of a cycle of length $n$ where if $n$ is even, each vertex is connected to the one directly opposite it by a chord, or, if $n$ is odd, to the two vertices 'almost opposite' it. This definition, which Guy and Harary used, is visually descriptive, but a mathematical definition will be more helpful. In the even case, if we label vertices 0 through $2r-1$, edges adjoin numerically adjacent vertices, or vertices that differ by $r$ (modulo $2r$). In the odd case with the same labelling, edges adjoin numerically adjacent vertices and vertices that differ by either $r$ or $r+1$ (again, modulo $2r$). Figure 4 depicts $M_8$ and $M_9$, the Möbius ladders on eight and nine vertices, respectively.

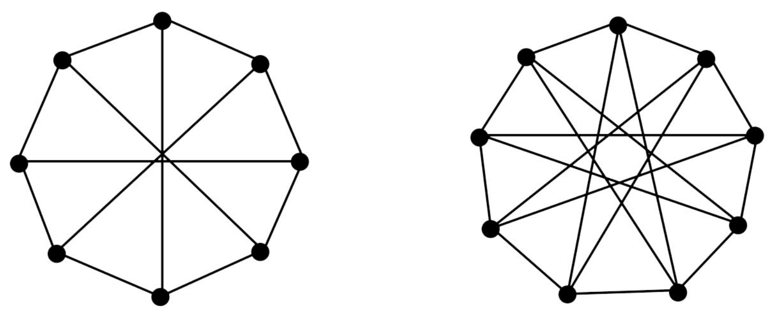

Figure 4: $M_8$ and $M_9$

## 2 Introduction to Topology

### 2.1 Planarity

Some properties of graphs are not independent of how they are drawn. For example, many graphs can be drawn in the plane in such a way that none of their edges intersect except at vertices. Such a drawing is called a **planar drawing**; a graph is called **planar** if there exists a planar drawing of it. A special type of planar graph is an **outerplanar** graph, which has an additional requirement that the vertices must be embedded in a circle with all the edges interior to it and not crossing. An outerplanar graph on $n$ vertices can contain at most $2n-3$ edges: $n$ of them form a cycle around the circle, and the remaining $n-3$ triangulate the interior. In lieu of a proof, see Figure 5 below.

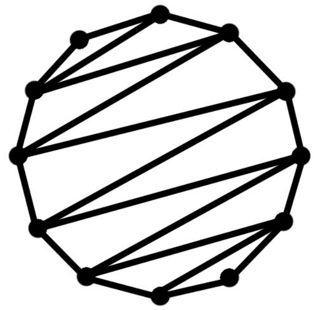

Figure 5: An outerplanar graph with the most possible edges on twelve vertices

Suppose we have a nonplanar graph $G$, and suppose that it is drawn (with crossing edges) on a plane. A **subgraph** of a graph $H$ is a graph that contains only vertices and edges from $H$. Notice that every graph is a trivial subgraph of itself. It should be clear that we can find some subgraph $G'$ of the planar nonplanar graph $G$ that is planar if we delete enough edges from $G$. If we then consider the remaining edges of $G$, we can find another planar subgraph $G''$. By continuing until all of the edges in $G$ have been associated with a planar subgraph, we end up with a representation of $G$ as a sort of 'stack' of planar graphs. Now suppose we are not constrained to a particular drawing of the original graph $G$ on the plane, but are allowed to embed it however we wish. We can then find an efficient way of dividing $G$ up into the fewest number of planar subgraphs possible. This number is called $G's$ **thickness**. Note that although we can draw $G$ however we wish, we cannot draw its subgraphs freely — when the subgraphs are super-imposed on one another their vertices must lie on top of each other. Figure 6 demonstrates that $K_{3,3}$ has thickness two, with the second subgraph indicated by dashed lines.

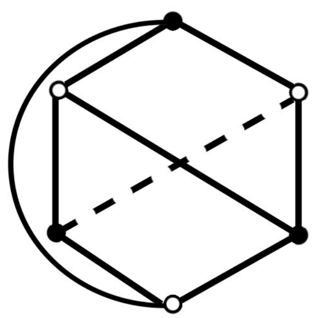

Figure 6: $K_{3,3}$ has thickness two

## 2.2 Higher genus surfaces

Let us consider a planar graph drawn on the plane again. A **face** is a region bounded by edges of the graph (see Figure 7). There is an easily overlooked face to every planar graph — the **infinite face** that is bounded by the graph's outer edges. There is nothing special about the infinite face; any face in the graph can be made into the infinite face by a redrawing. Consequently, it is reasonable to think of that outer face as not being infinite at all. Suppose we did not draw our graph on the plane, but on a sphere instead. Then the area of the finite face would be finite, since the surface of a sphere is finite. Any graph that can be drawn on the plane without edge crossings has a **two-cellular embedding** onto the sphere, meaning that every face is a polygon. Notice that the infinite face in an embedding onto the plane violates the two-cell requirement, because a polygon with finitely many edges cannot have infinite area.

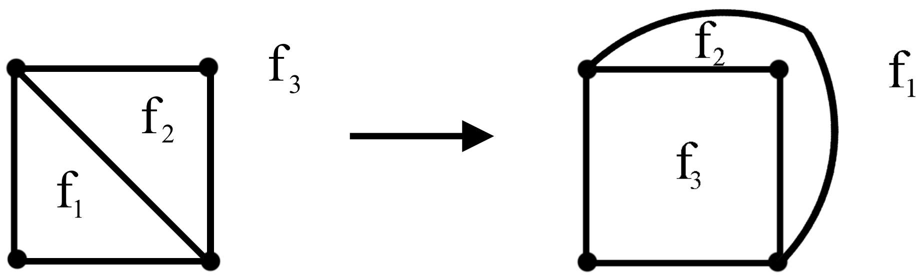


Figure 7: The infinite face is not unique

Now let us consider a graph that cannot be cellularly embedded onto the sphere. Suppose we can draw it in such a way that removing only one edge produces a planar drawing. If we could give that edge a bridge to travel on over the rest of the graph, then it would not cross any other edges, so let us imagine pulling a 'handle' out of the sphere and drawing the recalcitrant edge on it (see Figure 8). Now we have a sphere with a handle sticking out of it; if we were to distort the sphere portion we could make it look like a coffee cup. Instead, let us expand the handle until it is as big as the rest of the shape. Now we have a donut; the arch under the handle became a hole in the sphere. These three shapes are all in a way the 'same'–we did not add, twist, or break anything, we simply inflated or distorted it. In topological terms, we would say the shapes are **homeomorphic**[1].

Figure 8: Deformations of the handled sphere

The sphere with a hole in it is called a **torus**. We can stick as many holes in a sphere as we like (or add as many handles as we want) to create more surfaces. Graphs that cannot be embedded on a torus can be embedded on a sphere with more holes. I should note that we are considering only

[1]This is why mathematicians joke that a topologist is a person who cannot tell a coffee cup from a donut.

the surfaces of these various shapes, which is why we do not care what their volume or distortion is — we are considering them as two-dimensional rather than three-dimensional objects.

If we have a surface with a two-cell embedding of a graph on it, we can determine its **orientability**. A surface is orientable if it is possible to assign orientations (clockwise or counterclockwise) to the polygonal faces of the graph embedded onto it such that if two faces share an edge, then the faces induce opposite directions on that edge. So if one face is given a clockwise orientation, an adjacent face must also have a clockwise orientation. The sphere is an **orientable surface**, as are spheres with holes in them. Together, they form the family of all closed orientable surfaces, and we call the sphere with $g$ holes in it $S_g$, and call $g$ the **genus** of the surface $S_g$.

There are also nonorientable closed surfaces. Consider the Möbius strip. If you have never seen one before, it can be represented by taking a long strip of paper, twisting it, and taping the short edges together. Now imagine that the paper is truly two-dimensional and thererfore is a surface with no thickness, like a slice of a plane. The Möbius strip is, however, nonorientable, as we can see from Figure 9 (the arrows indicate that the short sides should be glued with a twist). The blank triangular face cannot be assigned an orientation. Notice also that the Möbius strip has one edge: if you run a finger along the 'top', you will return back to where you started having traversed the whole edge.

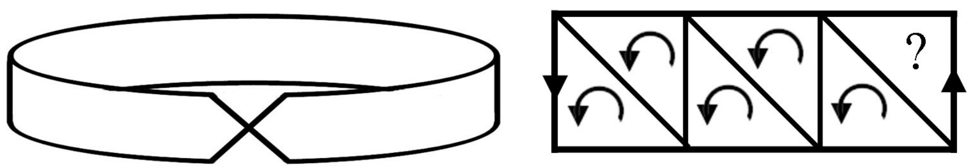


Figure 9: The Möbius strip is nonorientable.

Now consider again the sphere. Suppose this time we do not cut a hole all the way through it, but instead cut a hole out of its surface. Now we have a disk missing, so let us patch it with a Möbius strip by gluing its edge to the boundary of the missing disk. This is the **nonorientable surface with crosscap number one**, and is written $N_1$. If we cut $k$ disks out of the sphere and patch them with $k$ Möbius strips, the resulting surface is called $N_k$ and has crosscap number $k$. It is extraordinarily difficult to picture even $N_1$. Even the orientable surfaces, which are easy to imagine and draw, become difficult to picture with graphs embedded onto them. Consequently, we have developed several simpler ways of representing graphs embedded onto surfaces, one of which we will discuss now and two of which we will examine in Sections 6 and 7.

## 2.3 Embedding graphs onto surfaces

We already know that when we want to embed a graph onto the sphere we can draw it on a plane instead. More than that, we could draw it inside a rectangle.

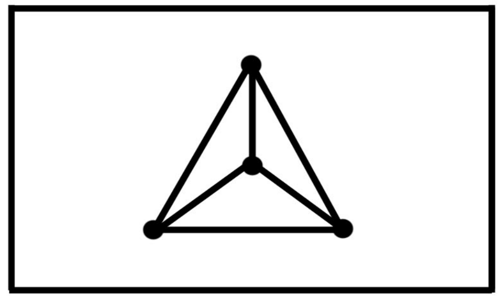

Figure 10: $K_4$ on the sphere.

Similarly, we can represent a torus as a rectangle, by identifying opposite edges. Think of taking the two long sides and gluing them together to form a cylinder, and then gluing the two short sides together to form a torus. Notice how edges that go 'out' one side come 'in' the opposite.

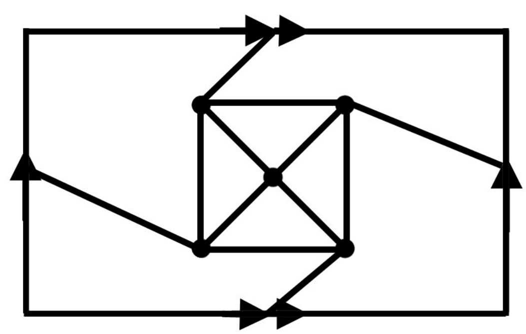

Figure 11: $K_5$ on the torus.

A Klein bottle, the nonorientable surface with crosscap number one, can also be drawn as a rectangle. This time, although we identify the same sides as we did for the torus, we put a twist on the short sides — notice how an edge going out the top of the right sides comes in at the bottom of the left.

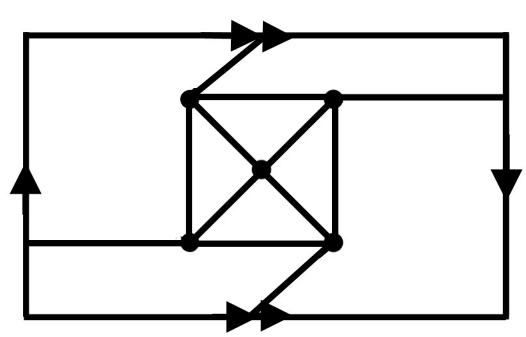

Figure 12: $K_5$ on the Klein bottle.

Now that we can work more comfortably with some of these surfaces, we can consider another property of graphs, their **genus** and their **crosscap number**. A graph is said to have genus $g$ if it has a two-cell embedding onto $S_g$ but cannot be embedded onto $S_k$ for every $k$ less than $g$. In fact, it is sufficient to say that the graph cannot be embedded onto $S_{g-1}$, because as we will discuss in Section 5, the surfaces graphs can embed onto come in intervals of integers. That is, if there exist two-cellular embeddings of $G$ onto $S_n$ and $S_{n+2}$ then there exists a two-cellular embedding of $G$ onto $S_{n+1}$. Similarly, a graph has crosscap number $k$ if it can be embedded onto $N_k$ but not $N_{k-1}$.

## 2.4 The Möbius ladder again

We conclude this section by returing to the Möbius ladder to see how planarity and genus apply to it. In their paper, Harary and Guy proved the surprising fact that any Möbius ladder has genus one. To do so, first they had to prove that it was not planar. $K_{3,3}$ is known to be nonplanar, and note from the solid lines on Figure 13 that every even-vertex Möbius ladder with six or more vertices contains $K_{3,3}$ as a subgraph. If a graph contains a nonplanar subgraph, then the graph must be nonplanar. It is more difficult to show that every odd-vertex Möbius ladder with five or more vertices are something called Kuratowski graphs, which follows from the fact that $M_5$ is the same as $K_5$. Kuratowski graphs are nonplanar. Therefore, all Möbius ladders are nonplanar.

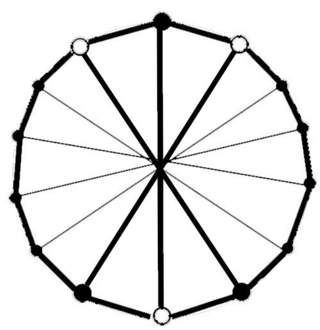

Figure 13: The Möbius ladder contains $K_{3,3}$.

The more interesting part is proving that all Möbius ladders can be embedded in the torus. Harary and Guy discovered the embeddings in Figure 14, which look like ladders that have been twisted into Möbius strips. They both have exactly one edge crossing, which Harary calls being "minimally nonplanar." Consequently, adding exactly one handle to the sphere will remove all crossings, and therefore all Möbius ladders can be embedded onto a torus. This is a surprising result, since their geometric construction involved many crossings.

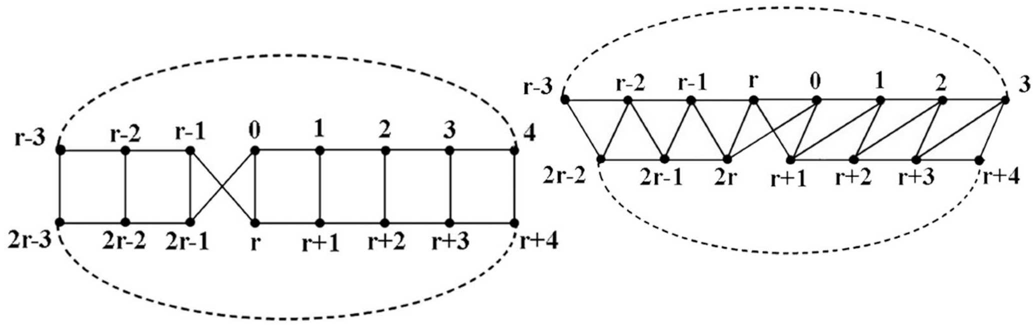


Figure 14: Layouts for even and odd Möbius ladders.

# 3 Bookthickness

In 1979, Frank Bernhart and Paul Kainen wrote a paper called "The Book Thickness of a Graph," in which they proposed a new surface for graph embeddings [1]. This surface was called a **book** because it could be visualized as several half-planes joined together along a line $L$, which looked rather like several pages joined together along a spine. Graphs are embedded onto a book by ordering their vertices along the 'spine' in some order $\sigma$ and then drawing their edges either on $L$, when the edge's vertices are sequential in $\sigma$, or on exactly one of the half plane 'pages'. The question Bernhart and Kainen sought to answer in their paper was a natural one: how many pages does a graph $G$ require in order to be embedded in a book with no edge crossings? They termed this number the graph's **book thickness**, although it has also been called 'bookthickness' and is now generally called 'pagenumber'. In this paper, we will denote a graph $G$'s bookthickness by $bt(G)$.

## 3.1 Origin of the book

The three simplest results from Kainen and Bernhart's paper are in the following theorem. A **connected graph** is one in which any two vertices are connected by a sequence of adjacent edges, none of which repeat, called a **path**.

**Theorem 3.1 ([1], Theorem 2.5)** *Let $G$ be a connected graph. Then*

*(i) The bookthickness of $G$ is zero if and only if $G$ is a path.*

*(ii) The bookthickness of $G$ is less than or equal to 1 if and only if $G$ is outerplanar.*

*(iii) The bookthickness of $G$ is less than or equal to 2 if and only if $G$ is a subgraph of a Hamiltonian planar graph.*

The first of these is obvious, because every edge can be embedded onto the spine and if the spine contains all edges and is not disconnected, it therefore contains a path. The second follows from an alternative representation of a book. Imagine one page of a book; we can imagine the spine and page to be finite because we are not considering infinite graphs. Then we can bend the spine around into a circle with all of the edges on the page in its interior. No space has been lost, and no crossings have been forced. Therefore, every page of a book is, when taken with the spine, an outerplanar graph (see Figure 15). In a way, 'bookthickness' is a specific type of thickness, where each subgraph has the additional restraint of being outerplanar.

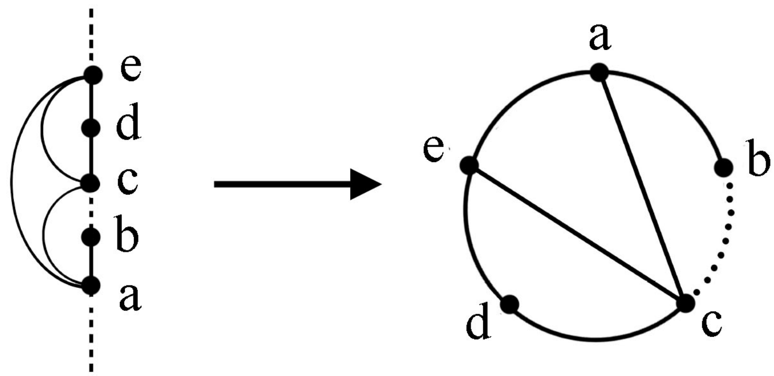


Figure 15: A page of a book can be drawn two ways.

The third property follows from a well-known phenomenon of a type of graph called Hamiltonian. A **cycle** is a path that begins and ends at the same vertex, and a cycle is called a **Hamiltonian cycle** if it includes every vertex exactly once except for the starting vertex, which it includes twice. A **Hamiltonian graph** is a graph containing a Hamiltonian cycle, and it has long been known that in a planar graph the edges interior to a Hamiltonian cycle form one outerplanar subgraph, and the edges exterior to the cycle form a second. The result (iii) follows immediately.

The next result is worth stating as a theorem, because it is so useful.

**Theorem 3.2 ([1] Theorem 3.3)** *Let $G$ be a graph with $n$ vertices and $m$ edges. Then $bt(G) \geq \lceil \frac{m-n}{n-3} \rceil$.*

**Proof** The spine can hold at most $n$ edges. Each page can hold at most $n-3$ edges, because each page is an outerplanar subgraph on $n$ vertices (Theorem 3.1). Therefore, $bt(G)(n-3)+n \geq m$, and the result follows immediately. **QED**

From this theorem, Kainen and Bernahrt proved that the bookthickness of the complete graph ($K_n$) is $\lceil \frac{n}{2} \rceil$, as demonstrated in Figure 16, wherein each page is 'full', or maximally outerplanar ([1] Theorem 3.4). Notice that we generally draw books as circular pages instead of three-dimensional objects because they are simpler both to render and read.

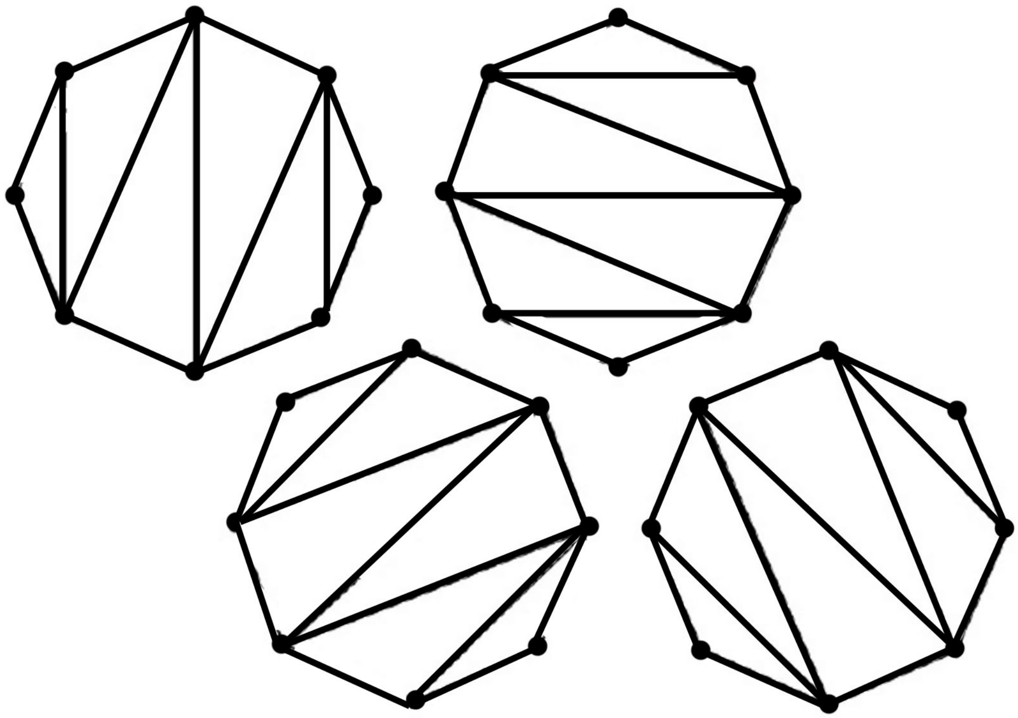

Figure 16: $K_8$ has bookthickness four.

## 3.2 Improvements

Several of Kainen and Bernhart's original theorems and conjectures were improved upon later. For example, they proved that $bt(K_{m,m})$ is less than or equal to $m$, which is a trivial result. When $m$ is greater than four, that bound can be improved to $m-1$. An upper bound for $K_{m,n}$, which follows from Theorem 3.2, is $bt(K_{m,n}) \geq \lceil \frac{mn-m-n}{m+n-3} \rceil$. These results were greatly improved in subsequent papers. Muder, Weaver, and West[9] lowered the upper bound for $bt(K_{m,n})$ in 1988 to min $\left\{\frac{2n+m}{4}, n\right\}$. Later, in 1997, Enomoto, Nakamigawa, and Ota [5] improved this bound further in for the case where $m = n$ to $bt(K_{n,n}) \leq \lfloor \frac{2n}{3} \rfloor + 1$.

Another characteristic of bookthickness that has been studied since Kainen's and Bernhart's first paper is the way in which genus relates to bookthickness. Bernhart and Kainen claimed that a planar graph could have arbitrarily high bookthickness, but in 1984, Buss and Shor [2] showed that all planar graphs could be embedded in at most 9 pages, which Heath [8] improved that same year to 7 pages. Finally, in 1986, Mithalis Yannakakis [11] proved that all planar graphs can be embedded in 4 pages, and provided a planar graph that required all four, proving his lower bound to be optimal. A famous result of graph theory states that the faces of any planar graph can be coloured with at most four colours so that no adjacent faces share a colour. Since any graph of genus one requires at most seven colours, it was conjectured that the bookthickness of a toroidal graph is at most seven. In 1997, T. Endo proved this conjecture [4]. This is especially interesting because neither proof used the four colour or seven colour theorems in their constructions. The numbers are the same, but apparently unrelated.

### 3.3 The Möbius ladder again again

Once again, we will conclude this section by asking how it relates to the Möbius ladder. We know that the Möbius ladder is not planar, so its bookthickness must be three or greater, from properties (ii) and (iii) of Theorem 3.1. Moreover, if we take the ladder-like layout that Guy and Harary designed and spread the vertices out so they look more like a circle, we can see a potentially helpful $\sigma$ for the vertices, pictured in Figure 17. In the even case, the straight vertical lines can all go on one page together, along with the edge $\{r-1, r\}$. The edge $\{0, 2r-1\}$ gets its own page, and the remaining horizontal lines get a third. Therefore, the bookthickness of a Möbius ladder with an even number of vertices is at most three.

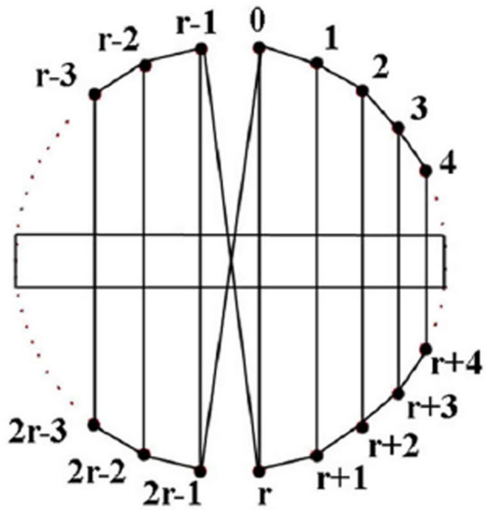


Figure 17: Layout for a book embedding of a Möbius ladder with an even number of vertices.

Again stealing Guy and Harary's layout, examine the case for a ladder with an odd number of vertices. This time, the zig-zag edges get the first page, while $\{0, 2r\}$ gets its own, and the remaining horizontal zig-zag edges get a third. Therefore, the bookthickness of any Möbius ladder is at most three, and since it is also at least three, we know that it is exactly three. It is gratifying that $M_n$ has a bookthickness that does not depend on $n$, since its genus does not either.

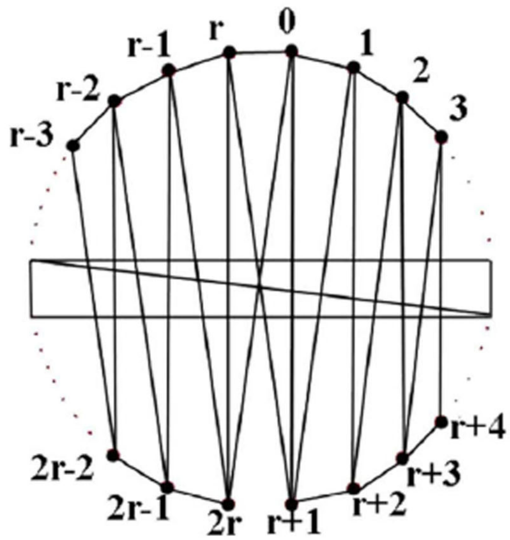


Figure 18: Layout for a book embedding of a Möbius ladder with an odd number of vertices.

# 4 The Bookthickness of Nets

As a result of a University of Puget Sound Summer Research Grant for Science or Mathematics, Summer of 2005, I was able to conduct original research into book embeddings. From this research, I developed a family of graphs I call a net, which can be constructed from outerplanar pages of a book or from the Möbius ladder. The interesting thing about nets is that half of them have an explicit bookthickness while the bookthickness of the other half is known only to a range, which is odd because the construction is the same in both cases. Let us begin by defining a net, and explaining my motivation for developing them.

## 4.1 Net construction

Recall the book embedding of a complete graph, and how each page (except for the last, in the case of an odd number of vertices) is triangulated. If we consider the spine as part of every page, then each page is maximally outerplanar. An interesting exercise is to consider what happens when we take these fully triangulated pages and make each pair of internal triangular faces into one quadrilateral face instead. To put it another way, we remove the diagonal lines from the interior of the page. We now have a graph constructed of layers of parallel lines and connected around the edge by a Hamiltonian cycle. Because this looks like a mesh of parallel strands with a boundary, I call the resulting graph a net.

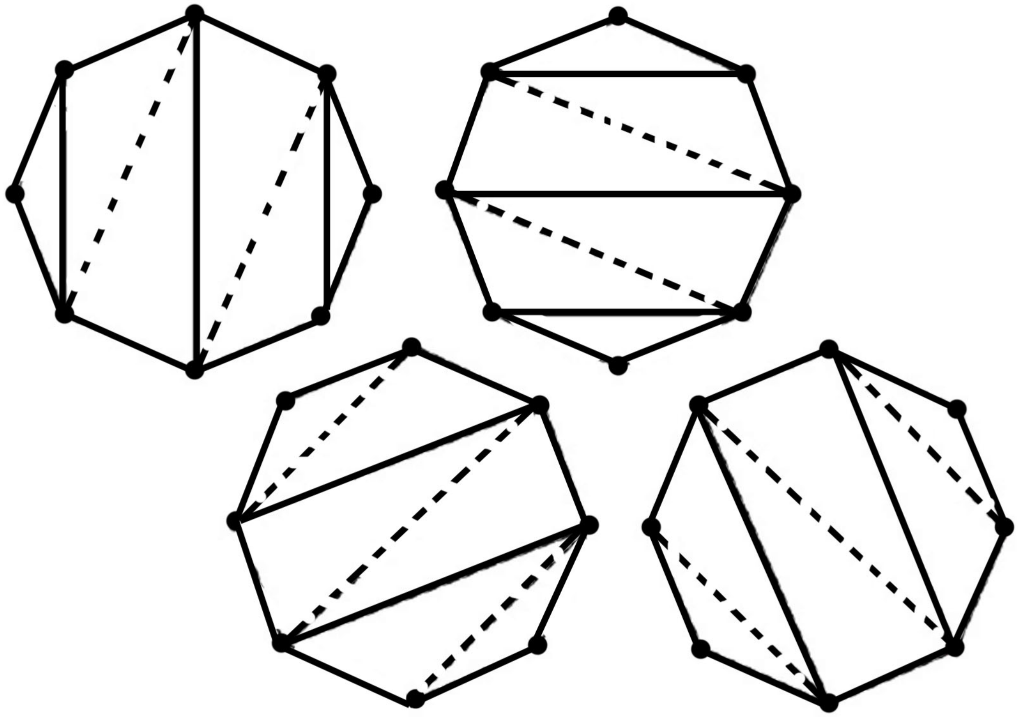

Figure 19: Making quadrilaterals out of outerplanar pages.

Although this is a simple geometric description of a net, a more mathematical one is desirable. Recall the Möbius ladder, which had a Hamiltonian cycle and connected 'opposite' vertices. If we instead think of the net as adding in every edge that is parallel to a central strut of a Möbius ladder, we can redefine it. Notice that when the number of vertices is odd, a Möbius ladder connects 'nearly opposite' vertices, and so this construction will only work for even numbers of vertices. The following definition is reminiscent of our precise definition of Möbius ladders and relies on vertex labels. $E(G)$ denotes the edge set of a graph $G$.

**Definition 4.1** *A **net**, $N_{2r}$, is a graph on $2r$ vertices, labeled $0$ to $2r-1$. All addition is modulo $2r$.*

*Define*

$$P_j = \left\{ \{k+j, r-k+j\} | 1 - \left\lceil \frac{r}{2} \right\rceil \leq k \leq \left\lceil \frac{r}{2} \right\rceil - 1 \right\}$$

*The edge set of $N_{2r}$ is*

$$EN_{2r} = \bigcup_{j=0}^{r-1} P_j \bigcup \{\{v, v+1\} | 0 \leq 2r-1\}$$

Graphically, compare a page from $K_{12}$ to $P_0$ from $N_{12}$:

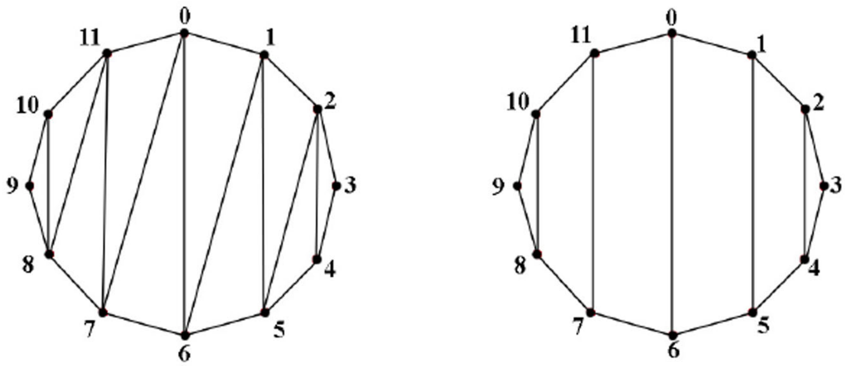


Figure 20: A page from a book embedding of $K_{12}$ and $P_0$ from $N_{12}$.

Since we created $N_{2r}$ by stripping edges off of $K_{2r}$ and adding edges to $M_{2r}$, the Möbius ladder on $2r$ vertices, it is immediately obvious that $bt(M_{2r}) \leq bt(N_{2r}) \leq bt(K_{2r})$, which means $3 \leq bt(N_{2r}) \leq r$. This range can be made arbitrarily large by increasing $r$, so we must find yet another way of describing $N_{2r}$. The following theorem claims that a net on $2r$ vertices, when $r$ is even, is similar to a pair of disjoint complete graphs on $r$ vertices, joined together by a Hamiltonian cycle. In the proof below, $|E(G)|$ denotes the size of the edge set of a graph $G$.

**Theorem 4.1** *When $r$ is even, and $H$ is the Hamiltonian cycle $0, 1, 2, ..., 2r-1$, then $EN_{2r} \setminus EH = E(2K_r)$.*

**Proof** Assume $N_{2r}$ is labeled 0 through $2r-1$ and that all arithmetic is modulo $2r$.

Examine some vertex $j$. In $P_j$, $j$ is incident to the edge $\{j, r+j\}$. By hypothesis, $r$ is even, so $j$ and $r+j$ have the same parity. In $P_{j+i}$, $j$ is incident to the edge $\{j, r+j+2i\}$. Then for any $P_i$, $j$ is connected to a vertex whose label has the same parity, and so

$$EN_{2r} - EH \subset E(2K_r).$$

where the even vertices and adjacent edges form one copy of $K_r$ and the odd vertices and adjacent edges form the other.

Now, $N_{2r}$ contains $r-1$ edges for each $P_j$, of which their are $r$, and a further $2r$ edges in the Hamiltonian cycle, so $|EN_{2r}| - |EH| = ((r-1)r + 2r) - 2r = r^2 - r$ when r is even, and $|E(2K_r)| = 2\frac{r(r-1)}{2} = r^2 - r$, the sets $EN_{2r} - EH$ and $E(2K_r)$ have the same size. Therefore, $EN_{2r} - EH = E(2K_r)$.

**QED**

When $r$ is odd, however, the net on $2r$ vertices instead resembles the complete bipartite graph $K_{r,r}$, as the following theorem claims.

**Theorem 4.2** *When $r$ is odd, $EN_{2r} = EK_{r,r}$.*

**Proof** Examine some vertex $j$. Again, all addition is modulo $2r$. By the argument in the previous proof, $j$ is incident to any edge of the form $\{j, r+j+2i\}$ for $i$ and integer. Since $r$ is by hypothesis odd, $j$ and $r+j+2i$ have different parity. Consequently, $j$ is connected only to vertices whose labels do not have the same parity as $j$. Therefore, the graph formed by

$$\bigcup_{j=0}^{r-1} P_j$$

is bipartite, where the bipartite sets are determined by the parity of the vertex labels, and so $EN_{2r} \subset EK_{r,r}$.

If we disregard edges in $P_j$ that are contained in the outer Hamiltonian cycle, $N_{2r}$ contains $r-2$ edges for each $P_j$, of which there are $r$, and $2r$ edges in the Hamiltonaian cycle. Therefore, $|EN_{2r}| = (r-2)r + 2r = r^2$ when $r$ is odd, and $|EK_{r,r}| = r^2$, and so the sets $EN_{2r}$ and $EK_{r,r}$ have the same size. Therefore, $EN_{2r} = EK_{r,r}$.

**QED**

## 4.2 The bookthickness of a net

It is interesting that such different graphs arise from the same construction, depending only on the parity of $r$. It is especially so, since the bookthickness of $K_n$ is known, while the bookthickness of $K_{n,n}$ is known only up to a range, as I have discussed. Consequently, the bookthickness of $N_{2r}$ is easy to find when $r$ is even.

**Theorem 4.3** *When $r$ is even, $bt(N_{2r}) \leq \frac{r}{2} + 1$*

**Proof** Arrange the vertices into an order $\sigma$ so that all the evenly labelled vertices are in ascending numerical order, followed by all of the oddly labelled vertices in descending numerical order. Then the two copies of $K_r$ have no overlapping edges between them. We can therefore embed them both in the usual way for $K_r$, which requires $\frac{r}{2}$ pages (each of which has two copies of the usual page for $K_r$). The Hamiltonian cycle can be embedded onto its own page, except for the edge $\{2r-1, 0\}$, which can be embedded onto any of the other pages.

**QED**

Figure 21 shows an example of a net being given the above embedding. The Hamiltonian cycle is indicated by lines.

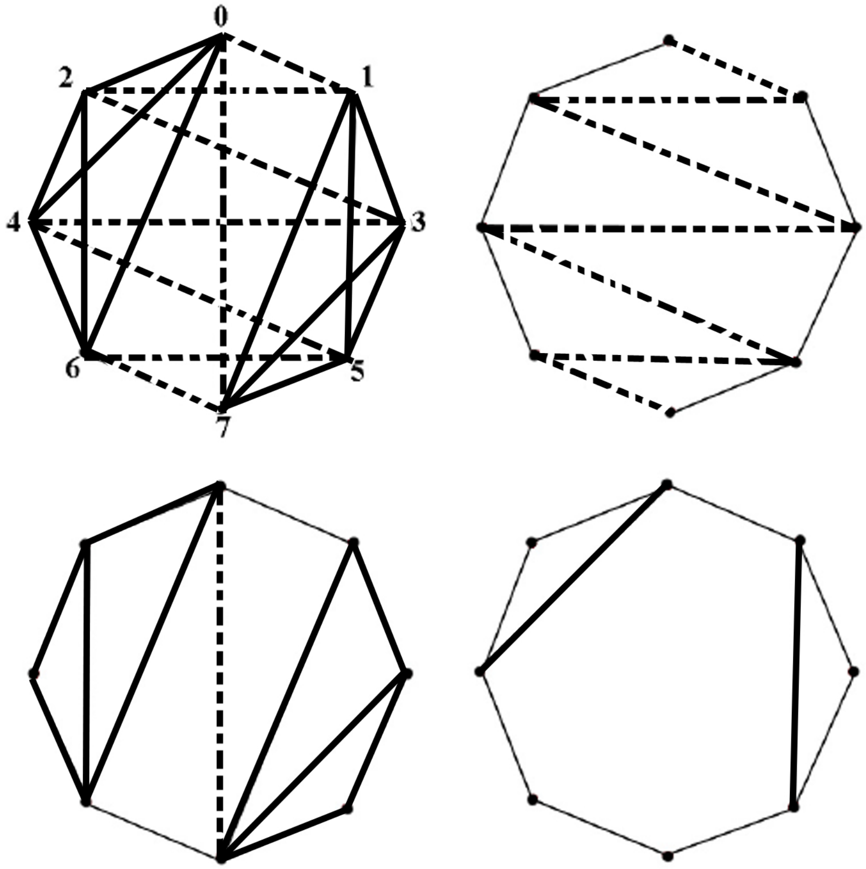


Figure 21: $N_8$ (top left) and its three page embedding.

This upper bound proves also to be the upper bound, resulting in the following theorem.

**Theorem 4.4** *$bt(N_{2r}) = \frac{r}{2} + 1$ when $r$ is even.*

**Proof** We already know that when $r$ is even, $bt(N_{2r}) \leq \frac{r}{2} + 1$. When $r$ is even, there are $2r$ edges in the Hamiltonian cycle of $N_{2r}$ and $r-1$ edges in $P_j$ for $1 \leq j \leq r$. Therefore, $|E(N_{2r})| = 2r + r(r-1) = r^2 + r$.

Suppose $bt(N_{2r}) \leq \frac{r}{2}$. Then by theorem 3.2, since $N_{2r}$ has $2r$ vertices and $r^2 + r$ edges,

$$\begin{aligned} r^2 + r = |E| &\leq \frac{r}{2}(|V| - 3) + |V| \\ &= \frac{r}{2}(2r - 3) + 2r \\ &= r^2 + \frac{r}{2} \end{aligned}$$

Which is a contradiction. Therefore, $bt(N_{2r}) = \frac{r}{2} + 1$.
**QED**

When $r$ is odd, $bt(N_{2r}) = bt(K_{r,r})$, and consequently finding its bookthickness is much harder. If we re-examine the lower bound found by Kainen and Bernhart, $\frac{r^2-2r}{2r-3}$, we will discover that the lower bound on both the even and odd case is the same:

**Theorem 4.5** *For $r$ an odd integer greater than 3,* $\left\lceil \frac{r^2-2r}{2r-3} \right\rceil = \left\lfloor \frac{r}{2} \right\rfloor + 1$.

**Proof** Since $r$ is odd and greater than 3, $r = 2k+1$ for some integer $k$ greater than 1, so

$$\begin{aligned}
\left\lceil \frac{r^2-2r}{2r-3} \right\rceil &= \left\lceil \frac{4k^2+4k+1-(4k+2)}{4k+2-3} \right\rceil \\
&= \left\lceil \frac{4k^2-1}{4k-1} \right\rceil \\
&= \left\lceil k + \frac{k-1}{4k-1} \right\rceil \\
&= k + \left\lceil \frac{k-1}{4k-1} \right\rceil \\
&= k+1
\end{aligned}$$

because $k-1 \leq 4k-1$ implies $\frac{k-1}{4k-1} \leq 1$.
Similarly, $\left\lceil \frac{r}{2} \right\rceil = \left\lceil \frac{2k+1}{2} \right\rceil = k+1$. Therefore, $\left\lceil \frac{k^2-2k}{2k-3} \right\rceil = \left\lceil \frac{r}{2} \right\rceil$, and because $r$ is odd, $\left\lceil \frac{r}{2} \right\rceil = \left\lfloor \frac{r}{2} \right\rfloor + 1$.
**QED**

Despite being able to give $N_{2r}$ the same lower bound for all cases, we are still unable to give it a definite bookthickness for the odd case. Doing so would mean we knew the bookthickness for a specific type of complete bipartite graph, and may enable us to draw conclusions about the bookthickness of other complete bipartite graphs.

# 5 Rotation Systems

We have discussed embedding graphs onto closed orientable and nonorientable surfaces, and how to represent these surfaces by means of polygons with identified sides. For a torus or a Klein bottle, this polygonal representation is simple, but for higher genus and higher crosscap surfaces it can become unruly. In this section, we will discuss another method of drawing graphs embedded onto particular surfaces that does not become unruly so quickly. Moreover, it can be succinctly represented in a tabular form known as a rotation system.

## 5.1 A new drawing method

Recall from section 2.2 that in order for a graph to be properly embedded into a surface it must be a cellular embedding. That is, each face must be homeomorphic to a disk. In Figure 22, we show $K_5$ embedded onto a Klein bottle, using the polygon representation. Next to it, we show the same embedding but without explicitly representing the surface, making sure that the edges adjacent to each vertex are in the same order that they were on the Klein bottle. The edge that passes through the nonorientable part of the Klein bottle gets an $x$ on it to indicate that it is 'twisted', and we call it a 'type-1 edge'. There is enough information in the second picture to deduce what surface is being used, but before we learn how to make that deduction, we must state some theorems.

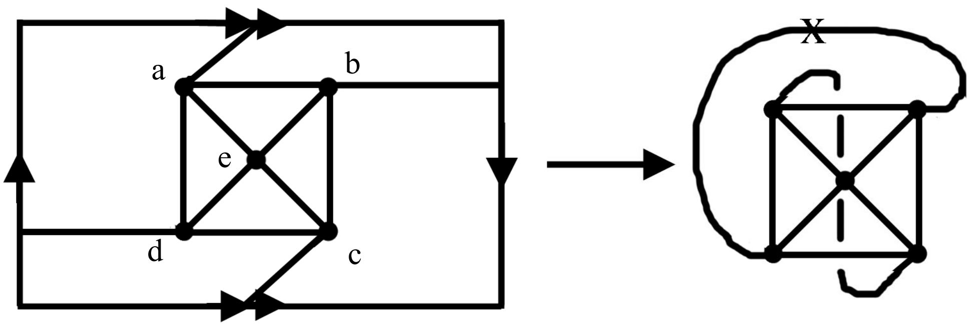


Figure 22: Two ways to draw the same embedding of $K_5$ onto the Klein bottle.

**Theorem 5.1 ([6] Theorem 3.3.3)** *(The Euler characteristic for orientable surfaces) Let $n$ be the number of vertices in a graph $G$, and $m$ be the number of edges. Then, if $G$ has a cellular embedding with $f$ faces onto an orientable surface $S_g$, $n - m + f = 2 - 2g$.*

**Theorem 5.2 ([6] Theorem 3.3.4)** *(The Euler characteristic for nonorientable surfaces) Let $n$ be the number of vertices in a graph $G$, and $m$ be the number of edges. Then, if $G$ has a cellular embedding with $f$ faces onto a nonorientable surface $N_k$, $n - m + f = 2 - k$.*

These two theorems follow from the Euler characteristic for the sphere, which was originally stated by Euler in 1750 as a formula relating the numbers of vertices, edges, and faces of a polyhedron, namely $n - m + f = 2$. Recall that the genus of a sphere is zero. We shall not go into the proofs here, although they can be found in [6]. This means that if we know how many edges and vertices a graph $G$ has, and how many faces an embedding of it into an unknown closed surface has, we can determine what that surface is. Returning to our new drawing method from above, all we need to do in order to discover what surface $G$ is embedded onto is find out how many faces are contained in the embedding.

Consider once again the polygon representation of $K_5$ embedded onto the Klein bottle. We can trace out the faces, as shown below, by following a walk around the boundary of the face. Notice that when we reach a vertex, we move on from the edge we were following to the edge next to it. There is no reason that we could not do this on the alternate representation, because we were careful to keep the edges in the same order on each vertex. Figure 23 also shows the same face tracing on the alternate representation.

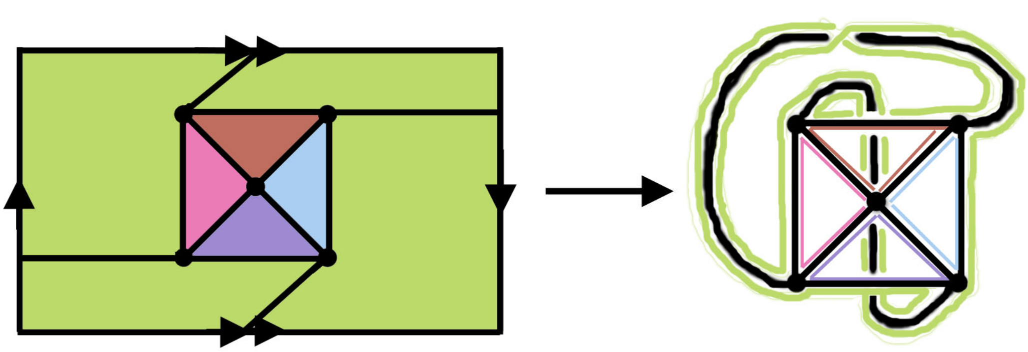

Figure 23: Faces traced out on the $K_5$ embedding onto the Klein bottle.

We can formalize this process into the **Face Tracing Algorithm**, once we create a tabular representation of our new drawing, called a **rotation system**. First, list each vertex by name. Second, for each vertex write the edge adjacent to it in clockwise order. If an edge is type-1, add a superscripted '1' above its name. Alternately, instead of naming each edge, we can write the name of the other vertex adjoining that edge. For example, the embedding of $K_5$ onto the Klein bottle from Fig 5.1 can be represented by the following rotation system:

a. bedc
b. d$^1$cea
c. adeb
d. cb$^1$ae
e. bcda

We can now use the face tracing algorithm outlined by Gross and Tucker to determine a face by tracing out its boudnary [6]:

1. Choose an initial verex $v_0$ and an initial edge $e_1$ incident on $v_0$. Let $v_1$ denote the other endpoint of $e_1$.

2. If the number of type-1 edges traversed so far is even, let $v_2$ denote the vertex in the rotation system at $v_1$ that comes after $v_0$. If the number of type-1 edges traversed so far is odd, let $v_2$ denote the vertex in the rotation system at $v_1$ that comes before $v_0$. Let $e_2$ denote the edge $\{v_1, v_2\}$.

3. Continue in this manner until edge $e_n$ such that the next two edges in the boundary would be $e_1$ and $e_2$.

If we apply the face tracing algorithm to the rotation system for $K_5$ above, choosing $v_0$ to be $a$ and $e_1$ to be $\{a, c\}$, we get a walk that traverses the following vertices in order: $a, c, d, b, a, c, b, d, a, c, d$. Notice that we passed through type-1 edges twice, and so each time reversed the direction in which we chose vertices.

## 5.2 Intervals of genus and crosscap number

Now that we have a tabular representation of a graph $G$ embedded into a particular surface, we can prove that the genus and crosscap numbers of surfaces into which some graph *can* be embedded comes in an interval of integers. First, define the embeddings $S$ and $T$ of a graph $G$ onto surfaces to be **adjacent** if there is an edge $e$ in $G$ such that deleting $e$ from both embeddings makes them identical. In other words, $S$ and $T$ differ by the placement of only one edge. Using our rotation systems, two embeddings are adjacent if deleting $e$ from their rotation systems results in identical rotation systems. Notice that the number of faces in embedding $S$ is at most one more or less than the number of faces in the embedding $S - e$. Therefore, $S$ has at most two more or fewer faces than $T$, which means their Euler characteristics differ by at most two, and so if they are both orientable surfaces they either have the same genus or the genus of one is one less than the genus of the other.

Using this information, Richard A. Duke proved the next theorem.

**Theorem 5.3 ([3] Orientable Interpolation Theorem)** *The genus range of a graph $G$ is an interval of continuous integers*

**Proof** ([6], Theorem 3.4.1)

If $L_1$ and $L_2$ are two rotation systems of a graph $G$, we can obtain $L_2$ from $L_1$ by permuting the entries for various vertices. Every such permutation can be written as a sequence of permutations that change only one edge in the whole rotation system. These consecutive rotation systems each represent adjacent cellular embeddings of $G$. Rotation systems that are adjacent, consequently, have genera that differ by at most one, giving us our conclusion.

**QED**

Similarly, the crosscap range of a graph is an unbroken interval of integers, which was proved by Stahl in 1978 [10]. This proof relies on more complicated techniques, which are beyond the scope of this paper.

# 6 Band Decompositions

We saw how we can encode an embedding onto a surface in a drawing of the graph using rotation systems. Now, we will go the other way around — taking a drawing of a graph, we will create a surface from it. This is similar to the spirit of rotation systems, but it is more visual and less tabular.

## 6.1 Making a band decomposition

Let us consider a drawing of $K_5$ again. A **1-band** is a topological space homeomorphic to the interval $[0,1] \times [0,1]$. In other words, it resembles a rectangle of some sort. We will call the arcs $[0,1] \times \{j\}$, for $j = 0, 1$, the 'ends' of the band, and the arcs $j \times [0,1]$ the 'edges' of the band. A **0-band** and a **2-band** are topological spaces homeomorphic to the unit disk. We can then define a **band decomposition** of a surface.

**Definition 6.1 ([6])** *A band decomposition of a surface $S$ is a collection of 0-bands, 1-bands, and 2-bands such that*

*(i) different bands intersect only along arcs in their boundaries.*

*(ii) the union of all the bands is $S$, each end of every 1-band is contained in the boundary of a 0-band.*

*(iii) each edge of every 1-band is contained in the boundary of a 2-band.*

*(iv) the 0-bands and 2-bands are pairwise disjoint.*

To state it more simply, the surface is composed of strips and two sets of disks so that every strip has its 'small' ends connected to disks from one set and its 'long' edges connected to disks from the other set. Additionally, no two disks from the same set intersect, and there are no overlaps between strips and disks except at their boundaries. Figure 24 is an example of a portion of a band decomposition of a surface. Notice that one edge has a twist in it–this is analogous to the type-1 edges we saw in the previous section. In general, we do not draw in the 2-bands, but let it be assumed that the gaps left by 0-bands and 1-bands are filled with 2-bands.

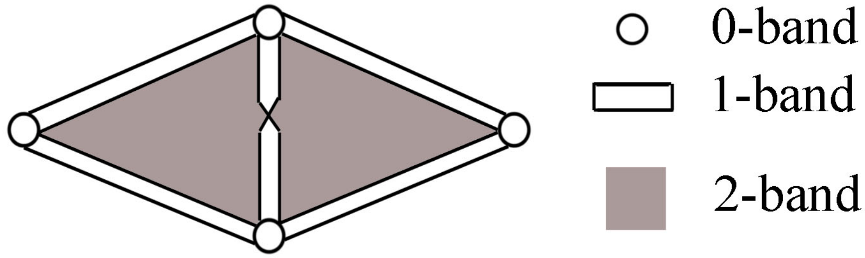


Figure 24: A portion of a band decomposition.

We can also think of this band-decomposition method as simply giving a drawn graph thickness. Each 1-band is a thickened edge, each 0-band a thickened vertex, and 2-bands are faces. In that way, the band decomposition forms a surface by 'filling in' the space between edges and vertices. Just as the rotation system and face tracing algorithm let us determine what surface the graph had been embedded onto, the band-decomposition in fact forms the surface.

## 6.2 Orientability

Now, let us consider the orientability of a band decomposition. We will assign each band of a band decomposition an orientation: clockwise or counter-clockwise. If two bands share a boundary, then on either side of the boundary the orientation must be going in the opposite direction in order for the surface to be considered 'orientable'. For example, if we have a 0-band with clockwise orientation and a 1-band whose boundary intersects that of the 0-band, and the 1-band is not twisted, we should give the 1-band a clockwise orientation as well. See Figure 25 below for examples. Now, take a band decomposition and assign every 0-band some orientation. A 1-band is called 'orientation preserving' if it is possible to assign it an orientation consistent with the orientation on the adjacent 0-bands. An edge is type-0 if its corresponding 1-band is orientation preserving, and type-1 else.

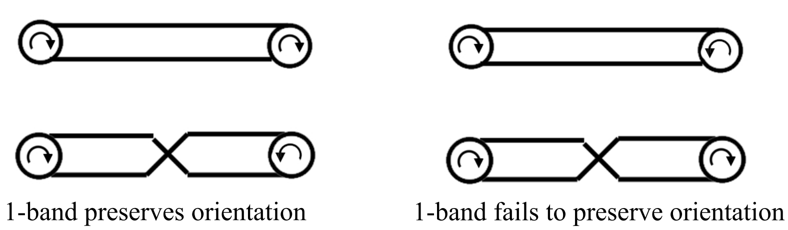


Figure 25: Orientation preserving and non-orientation preserving 1-bands.

Now, the surface created out of a graph by a band decomposition will be nonorientable if and only if there is a cycle in the associated graph that has an odd number of type-1 edges in it. This is consistent with our explanation in Section 2 of nonorientable surfaces, because such a cycle would form a Möbius band if we were to cut it out of the surface. If the cycle had an even number of type-1 edges, then its band decomposition would have an even number of twists in it. Consequently, the band could be 'untwisted'. If it has an odd number of twists in it, however, then we cannot untwist it completely. Think of a band that has two twists in it and convince yourself that it is functionally the same as a band with no twists.

This gives us a fairly simple algorithm for determining the orientability of a band-decomposition of a graph. First, define a **tree** to be a graph containing no cycles of any size. A **spanning graph** $S$ of a graph $G$ is a subgraph that contains every vertex in $G$ such that for any two vertices in $G$ there exists a path in $S$ from one to the other. A **spanning tree** is simply a spanning graph that is also a tree. Now we can use the orientability algorithm presented in [6]:

1. Choose a spanning tree $T$ from the graph $G$ whose band decomposition you are examining.

2. Next, choose a vertex $u$ from $T$ and give the corresponding 0-band an orientation.

3. Now, for every vertex connected to $u$ by an edge in $T$, choose an orientation such that the connecting 1-bands are orientation preserving.

4. Continue this process until every vertex has been assigned an orientation. The spanning tree will consist solely of type-0 edges.

A graph $G$ is a tree if and only if adding any edge to $T$ will form a cycle. Consequently, if there is an edge in $G$ but not $T$ that will be type-1 from the orientations assigned to the 0-bands corresponding to its vertices, that edge will form a type-1 cycle in $G$.

5. So the surface is nonorientable if and only if some edge in $G$, but not in $T$, is type-1.

Here is a step-by-step diagram of how we would apply this algorithm to a graph. The numerical labels correspond to the steps in the algorithm with the same labels.

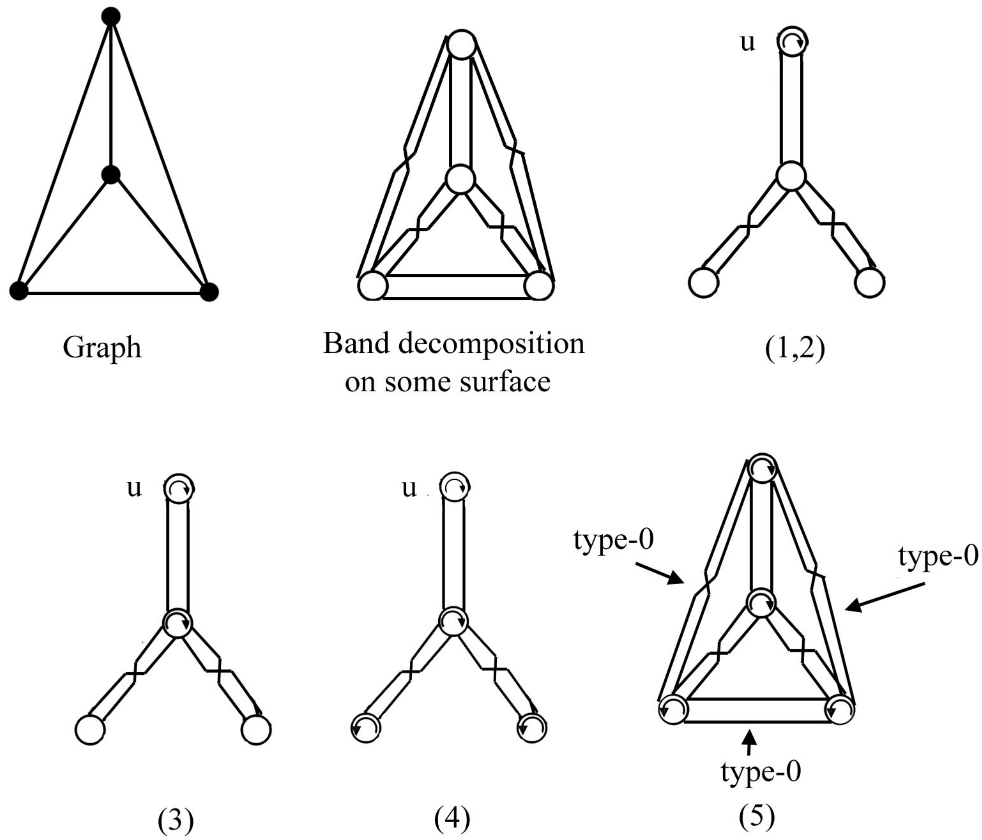


Figure 26: The face-tracing algorithm

# 7 Conclusion

We have now examined several surfaces onto which graphs can be embedded, and a few ways such embeddings can be represented. There are also several questions still to be answered about nets and their bookthickness. A possible subject for further research could be the bookthickness of a graph's complement, since the complement of $N_{2r}$ is closely related to $N_{2r+2}$, and a possible inductive argument could be formed. The bookthickness of a graph's complement has also not yet been studied and would be worth pursuing for its own sake.

It may also be possible to generalise books in the way that the torus and the Klein bottle were generalised into $S_g$ and $N_k$. One way of doing this could be to recognise that a book can be drawn as a set of nested spheres with their equators identified. Nesting tori together and identifying them at a boundary — perhaps a noncontractable curve around the hole, or a curve that cuts through both the outside and the hole — could produce a higher-genus book. It is less clear how to cut a two-holed torus, or one of higher genus still. We could also generalise a book by noting that each page is a rectangle and identifying edges of that rectangle so that each page is a Klein bottle or a torus.

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