Mathematics 181A **Fall 2010**

FINAL EXAM

December 13, 2010

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Name

Technology used: ____ **Only write on one side of each page.**
Show all of your work. Calculators may be used for numerical calculations and answer checking only.

Do BOTH of the following.

1. [10, 10, 10 points] A round hole of radius $\sqrt{3}$ft is bored through the center of a solid sphere of radius 2 ft.

   (a) Set up integral(s) for the volume of material removed from the sphere using the method of slicing.

   (b) Set up integral(s) for the volume of material removed from the sphere using the method of cylindrical shells.

   (c) Evaluate one of a. or b. above.

2. [30 points] Determine the radius of convergence, the values of x for which the following series converges absolutely and the values of x for which the series converges conditionally. Show all work.

$$1 + \frac{(x+4)}{3 \cdot 2} + \frac{(x+4)^2}{3^2 \cdot 3} + \frac{(x+4)^3}{3^3 \cdot 4} + \frac{(x+4)^4}{3^4 \cdot 5} + \cdots .$$

Do TWO (2) of the following.

1. [15 points] Use the First Fundamental Theorem of Calculus to determine the derivative of the function

$$F(x) = \int_x^{x^3} e^t \sqrt{3+t}\, dt$$

2. [15 points] Part 1 of the fundamental theorem of calculus tells us that if f is a continuous function on the interval $[a, b]$, then $\frac{d}{dx} \int_a^x f(t)\ dt = f(x)$. Explain both why

$$\frac{d}{dx} \int_0^x e^x\ dt = e^x + xe^x \quad \text{instead of} \quad e^x.$$

3. [4, 4, 4, 3 points] Use Riemann sums to carefully and fully explain why $\lim_{n\to\infty} \frac{\sqrt{3}+\sqrt{6}+\sqrt{9}+\cdots+\sqrt{3n}}{n^{3/2}}$ is equal to $\int_0^1 \sqrt{3x}\, dx$. In particular:

   (a) What is the value of Δx? (use the fact that the subintervals for the given Riemann sum are all equal in size.)

   (b) What are the values of the x_k in the partition $P = \{x_0, x_1, x_2, \cdots x_n\}$?

   (c) what are the values of the points c_k, $k = 1, 2, 3, \cdots, n$?

   (d) Why does $\|P\|$ limit to zero as n goes to infinity?

Do FOUR of the following.

1. [20 points each]

   (a) Evaluate
   $$\int \frac{x^3+3x^2-26x-41}{x^2+3x-28}\,dx$$

   (b) Evaluate
   $$\int \frac{dx}{\sqrt{2x-x^2}}$$

   (c) Evaluate
   $$\int_{-1}^{1} \frac{t+1}{\sqrt{t^2+2t}}\,dt$$

   (d) Does this improper integral converge or diverge? Why?
   $$\int_1^{\infty} \frac{1}{x^2\left(1+e^x\right)}dx$$

   (e) Find the exact sum of
   $$\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{3^{n-1}} + \sum_{n=1}^{3} 2^n$$

   (f) Does the series diverge, converge absolutely, or converge conditionally? Give complete explanations for your answers.
   $$\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$$

Do TWO (2) of the following

1. [10 points] Use a comparison test to prove: If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series of positive numbers, then $\sum_{n=1}^{\infty} a_n b_n$ must also converge.

2. [10 points] An important function in physics and statistics is the Gamma Function, $\Gamma(x)$, which has domain the set of all positive real numbers and is defined by
   $$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}\ dt.$$
   Use integration by parts to help you evaluate $\Gamma(3)$.

3. [5, 5 points] Use the Taylor Series for the standard functions listed in the Useful Information section at the end of this exam to do **both** of the following.

   (a) Find the Taylor series at $x=0$ and its interval of convergence for the function $f(x) = \frac{1}{1+x^3}$.

   (b) Evaluate the limit
   $$\lim_{h\to 0} \frac{\frac{\sin(h)}{h} - \cos(h)}{h^2}$$

Useful Information

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x|<1 \qquad \sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!}x^{2n+1}, \quad -\infty < x < \infty$$

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n, \quad -\infty < x < \infty \qquad \cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!}x^{2n}, \quad -\infty < x < \infty$$