Mathematics 290-A Exam 4 Fall 2006

November 14, 2006

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Name

Technology used: ______________________

- **Only write on one side of each page.**
- **Use terminology correctly.**
- **Partial credit is awarded for correct approaches so justify your steps.**
- **When given a choice, be sure to specify which problem(s) you want graded.**

Do any two (2) of these computational problems

C.1. Show that $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ is an eigenvector for the matrix $\begin{bmatrix} 2 & -6 & 6 \\ 1 & 9 & -6 \\ -2 & 16 & -13 \end{bmatrix}$ and determine the corresponding eigenvalue.

C.2. Given the subspace V of $\mathbf{C}^4$ where $V = \left\langle \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\rangle$, determine the dimension of the subspace $V^{\perp}$ by finding a basis for $V^{\perp}$.

C.3. The characteristic polynomial of $A = \begin{bmatrix} -2 & -6 & -6 \\ -3 & 2 & -2 \\ 3 & 2 & 6 \end{bmatrix}$ is $P_A(x) = -(x+2)(x-4)^2$. Find all eigenvalues and determine their algebraic and geometric multiplicities.

Do any two (2) of these problems from the text, homework, or class.

You may NOT just cite a theorem or result in the text. You must prove these results.

M.1. Prove Theorem RMRT, Rank of a Matrix is the Rank of the Transpose:
Suppose A is an $m \times n$ matrix. Then $r(A) = r(A^t)$.

M.2. From Project 11: Explain why the following 5×5 matrix that has a 3×3 zero submatrix is definitely singular (regardless of the 16 non-zeros marked by x's.)

$$\begin{bmatrix} x & x & x & x & x \\ x & x & x & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \end{bmatrix}$$

M.3. Exercise T60 in subsection PD (Properties of Dimension): Suppose that W is a vector space with dimension 5, and U and V are subspaces of W, each of dimension 3. Prove that $U \cap V$ contains a non-zero vector.

Do two (2) of these problems you've not seen before.

T.1. Label the following statements as being true or false.

(a) The rank of a matrix is equal to the number of its nonzero columns.

(b) The rank of a matrix is equal to the number of its nonzero rows.

(c) The $m \times n$ zero matrix is the only $m \times n$ matrix having rank 0.

(d) Elementary row operations preserve rank.

(e) An $n \times n$ matrix of rank n is invertible.

(f) It is possible for a 3×5 matrix to have rank 4.

(g) It is possible for a 5×3 matrix to have rank 4.

T.2. Suppose that A is a 4×4 matrix with exactly two distinct eigenvalues, 5 and -9 and let $E_A(5)$ and $E_A(-9)$ be the corresponding eigenspaces, respectively.

(a) Write all possible characteristic polynomials of A that are consistent with $\dim(E_A(5)) = 3$

(b) Write all possible characteristic polynomials of A that are consistent with $\dim(E_A(-9)) = 2$.

T.3. A matrix A is idempotent if $A^2 = A$. Show that the only possible eigenvalues of an idempotent matrix are $\lambda = 0$ and $\lambda = 1$. Then give an example of a matrix that is idempotent and has both of these two values as eigenvalues.

T.4. An $n \times n$ matrix A is **nilpotent** if, for some positive integer k, $A^k = O$, where O denotes the $n \times n$ zero matrix. Prove that if A is nilpotent, then A is not invertible.