Mathematics 181 **Spring 2007** **Quiz 5**

Due April 13, 2007

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Name

Only write on one side of each page.
Be sure to re-read the WRITING GUIDELINES rubric, since it defines how your project will be graded. In particular, you may discuss this project with others but **you may not collaborate on the written exposition of the solution.**
"It is hard to know what you are talking about in mathematics, yet no one questions the validity of what you say. There is no other realm of discourse half so queer." – J. R. Newman

The Background

In many ways, the 1600 and 1700's were the period when the ideas that underpin what we now call calculus were being developed. During this period there was a rich correspondence among those we would now call mathematicians, physicists or chemists. Often, one person would pose a question that he (or very rarely, she) had been studying and others would try to solve it. In some ways it was a kind of competition.
One such problem (which we might address later in the semester) was posed in 1644 by Pietro Mengoli (1625-1686). It was called the Basel problem and is easily stated: What is the sum of the reciprocals of the perfect squares? Today we would say: the following series converges by the integral test; to what sum does it converge?

$$\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

For nearly 100 years the best minds were not able to answer this question. However, around 1729, Leonard Euler found the sum. In fact, he found three different ways to compute it. One of those methods came from a very clever method for interpolating the outputs of functions that had integer domains. One of the functions he interpolated was the factorial function. What I mean by interpolation is this: factorials are defined only for nonnegative integers: $0! = 1,\ 1! = 1,\ 2! = 2 \cdot 1,\ 3! = 3 \cdot 2 \cdot 1$, etc. What Euler did was to find a function with domain the set of all real numbers except the nonnegative integers whose graph passes through all of the dots that are the graph of the factorial sequence $\{n!\}$. Here is a graph of a portion of this function.

Euler defined the output of his function given input m (which need not be an integer) as

$$\frac{1 \cdot 2^m}{1+m} \cdot \frac{2^{-m} \cdot 3^m}{2+m} \cdot \frac{3^{-m} \cdot 4^m}{3+m} \cdot \cdots . \tag{1}$$

Since we have not studied infinite products we will not use this form of Euler's function. Instead we will limit ourselves to the comment that the logarithm of this product would be an infinite series and perhaps what we are learning about series would allow us to fully understand what Euler was doing. Instead we will work with the modern formulation of Euler's function. It is called the **Gamma function** and is extremely useful in various applications in physics, statistics, and most places where factorials tend to crop up. As I noted above, this improper integral converges for every real number except the nonnegative integers and it's graph passes through the points $(n+1, n!)$ for every nonnegative integer n.

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\ dt.$$

Evaluating this function at different values of x is not trivial. For example, computing $\Gamma(2)$ is best done using integration by parts. As a start on the process, note that

$$\begin{aligned}\Gamma(2) &= \int_0^\infty t^{2-1}e^{-t}\ dt \\ &= \int_0^\infty te^{-t}\ dt.\end{aligned}$$

The purpose of this project is for you to explore some of the basic properties of the Gamma function using the integration tools we have developed.

The Project

1. As we explore the Gamma function, one of the tools we will need is the fact that if a is a positive constant then

$$\lim_{x\to\infty}\frac{x^a}{e^x} = 0.$$

   Prove this result by

   (a) Showing $\lim_{x\to\infty}\frac{x^n}{e^x} = 0$ for any positive integer n and

   Solution: When $n = 1$ we have $\lim_{x\to\infty}\frac{x^1}{e^x} = \lim_{x\to\infty}\frac{1}{e^x} = 0$ by L'Hospital's Rule so we know the formula works for at least one positive integer. But it is impossible for their to be a greatest positive integer for which the formula works because, if the formula works for the integer n (that is, $\lim_{x\to\infty}\frac{x^n}{e^x} = 0$) then, using L'Hostpital's Rule, we have $\lim_{x\to\infty}\frac{x^{n+1}}{e^x} = \lim_{x\to\infty}\frac{(n+1)x^n}{e^x} = (n+1)\lim_{x\to\infty}\frac{x^n}{e^x} = (n+1)(0) = 0$. The two facts that the formula holds for $n = 1$ and that there is not a largest integer for which it holds tells us that it must hold for **every** positive integer.

   (b) Using the sandwich theorem and part a.to show $\lim_{x\to\infty}\frac{x^a}{e^x} = 0$ for any positive constant a.

   Solution: Let n be a positive integer that is greater than a. Then $0 \le x^a \le x^n$ for all $x > 0$. This tells us that $\frac{0}{e^x} \le \frac{x^a}{e^x} \le \frac{x^n}{e^x}$. Since $\lim_{x\to\infty}\frac{0}{e^x} = \lim_{x\to\infty}(0) = 0$ and (by part 1.a) $\lim_{x\to\infty}\frac{x^n}{e^x} = 0$, the sandwich theorem tells us that $\lim_{x\to\infty}\frac{x^a}{e^x} = 0$ for any positive number a.

2. For this problem, you may use, without proving it, the fact that the improper integral $\int_0^\infty e^{-x^2}\,dx$ converges to the value $\sqrt{\pi}/2$. (This is usually shown in multivariable calculus and was known to Euler back in 1730.)

   (a) Directly evaluate the improper integrals for $\Gamma(1)$ and $\Gamma(2)$ using integration by parts.

   Solution:

   i. $\Gamma(1) = \int_0^\infty t^{1-1}e^{-t}\,dt = \lim_{b\to\infty}\int_0^b e^{-t}\,dt = \lim_{b\to\infty} -e^{-t}\big|_0^b = \lim_{b\to\infty}\left[\frac{-1}{e^b} - \left(\frac{-1}{e^0}\right)\right] = 0 + 1 = 1$
   $\Gamma(2) = \int_0^\infty t^{2-1}e^{-t}\,dt = \lim_{b\to\infty}\int_0^b te^{-t}\,dt$. Now, using integration by parts with $u = t$, $dv = e^{-t}dt$ so that $du = dt$ and $V = -e^{-t}$ we have

$\Gamma(2) = \lim_{b\to\infty} \int_0^b te^{-t}\,dt = \lim_{b\to\infty}\left[-\frac{t}{e^t} + \int_0^b e^{-t}\,dt\right] = -\lim_{b\to\infty}\frac{t}{e^t} + \lim_{b\to\infty}\int_0^b e^{-t}\,dt = 0 + \Gamma(1)$ by part 1.b. for the first limit and the definition of the gamma function for the second. Since we know that $\Gamma(1) = 1$ we conclude that $\Gamma(2) = 0 + \Gamma(1) = 1$.

(b) Use the substitution $t = u^2$ to evaluate the improper integral for $\Gamma(1/2)$.

Solution: $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{\frac{1}{2}-1}e^{-t}\,dt = \int_0^\infty t^{-1/2}e^{-t}\,dt$. Using the substitution $t = u^2$ so that $dt = 2udu$ we see that when $t = 0$ so does u and when t limits to infinity so does u. Hence, $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-1/2}e^{-t}\,dt = \int_0^\infty \left(u^2\right)^{-1/2} e^{-u^2}2udu = 2\int_0^\infty u^{-1}ue^{-u^2}\,du = 2\int_0^\infty e^{-u^2}\,du = 2\frac{\sqrt{\pi}}{2} = \sqrt{\pi}$ where the last equalities come from the given information about $\int_0^\infty e^{-u^2}\,du$.

(c) Use integration by parts to show that for any $x > 0$,

$$\Gamma(x+1) = x\Gamma(x) \tag{2.}$$

Solution: $\Gamma(x+1) = \int_0^\infty t^{(x+1)-1}e^{-t}\,dt = \int_0^\infty t^x e^{-t}\,dt$ and we use integration by parts with $u = t^x$, $dv = e^{-t}\,dt$ so that $du = xt^{x-1}dt$ and $V = -e^{-t}$.This tells us that $\Gamma(x+1) = \int_0^\infty t^x e^{-t}\,dt = \lim_{b\to\infty}\int_0^b t^x e^{-t}\,dt = \lim_{b\to\infty}\left[-\frac{t^x}{e^t} + x\int_0^b t^{x-1}e^{-t}\,dt\right]$.This last limit equals $\lim_{b\to\infty}\left(-\frac{t^x}{e^t}\right) + x\lim_{b\to\infty}\int_0^b t^{x-1}e^{-t}\,dt$ which equals $0 + x\Gamma(x)$ by using part 1.a. and the definition of the Gamma function.

i. Using your previously computed values, check that this formula gives you the proper value of $\Gamma(2)$.

Solution: Using the formula we obtain $\Gamma(2) = 1\Gamma(1) = 1 \cdot 1 = 1$ which agrees with our previous computation.

ii. Why can't we use $\Gamma(1)$ and this formula to tell us $\Gamma(0)$? Does your proof using integration by parts work for any other values of x besides when $x > 0$?

Solution:

A. The formula can be re-written as $\Gamma(x) = \frac{1}{x}\Gamma(x+1)$ telling us that we cannot compute $\Gamma(0)$ because it involves division by zero.

B. Our integration by parts proof of the formula cannot be extended to negative values of x because then the improper integral $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\ dt$ has **two** improprieties: one at 0 and the other at infinity and the one at 0 causes the integral to diverge. However, although integration by parts does not work, we can use the formula in displayed equation 2., $\Gamma(x+1) = x\Gamma(x)$, to extend the domain of Γ to those negative numbers that are not integers. For example, Since $\Gamma(x) = \frac{1}{x}\Gamma(x+1)$, we can **define** $\Gamma\left(-\frac{2}{3}\right) = \frac{1}{-2/3}\Gamma(1/3)$ and then use our work in part 2.c. to evaluate $\Gamma(1/3)$. In a similar manner we can define $\Gamma(x)$ for any negative number as long as none of $x+1, x+2, x+3, \cdots$ equals 0 since we cannot evaluate $\Gamma(0)$.

(d) **Without evaluating any more improper integrals,** use parts a., b., and c. to compute $\Gamma(2), \Gamma(3), \Gamma(3/2), \Gamma(5/2), \Gamma(-1/2)$, and $\Gamma(-3/2)$.

Solution: Using the formula we have

i. $\Gamma(2) = 1\Gamma(1) = (1)(1) = 1$ and $\Gamma(3) = 2\Gamma(2) = (2)(1) = 2$

ii. $\Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \sqrt{\pi}/2$ and $\Gamma(5/2) = \frac{3}{2}\Gamma(3/2) = \frac{3}{2}\left(\sqrt{\pi}/2\right) = 3\sqrt{\pi}/4$

iii. If we use formula $\Gamma(x) = \frac{1}{x}\Gamma(x+1)$ even when x is negative then $\Gamma(-1/2) = \frac{1}{-1/2}\Gamma(1/2) = -2\sqrt{\pi}$ and $\Gamma(-3/2) = \frac{1}{-3/2}\Gamma(-1/2) = -\frac{2}{3}\left(-2\sqrt{\pi}\right) = \frac{4}{3}\sqrt{\pi}$

(e) Explain why $\Gamma(n+1) = n!$ for every nonnegative integer n.

Solution: We already know that $\Gamma(1) = 1 = 0!$ and $\Gamma(2) = 1 = 1!$ so we know that the formula $\Gamma(n+1) = n!$ works for at least some positive integers. But there cannot be a greatest positive integer for which the formula holds (that is an integer for which every larger integer causes the formula to fail) because – if n is a positive integer for which the formula holds (that is, $\Gamma(n+1) = n!$) then $\Gamma(n+1+1) = (n+1)\Gamma(n+1) = (n+1)\,n! = (n+1)!$ This shows that there cannot be a largest positive integer for which the formula holds so, since it holds for $n = 0$ and $n = 1$, it must hold for every integer greater than or equal to 0.

(f) Use display 2 to explain why we can intuitively think of "$\left(\frac{1}{2}\right)!$"$= \Gamma\left(\frac{3}{2}\right)$ as the infinite product $\frac{2}{3} \cdot \frac{2}{5} \cdot \frac{2}{7} \cdot \frac{2}{9} \cdot \cdots$ and show that this agrees with Euler's formula in Display 1 when $m = \frac{1}{2}$.

Solution: Rewriting displayed formula 2., we have $\Gamma(x) = \frac{1}{x}\Gamma(x+1)$ being true for every positive number x. Hence we have

$$\begin{aligned} \Gamma\left(\frac{3}{2}\right) &= \frac{1}{3/2}\Gamma(5/2) \\ &= \frac{1}{3/2} \cdot \frac{1}{5/2}\Gamma(7/2) \\ &= \frac{1}{3/2} \cdot \frac{1}{5/2} \cdot \frac{1}{7/2}\Gamma(9/2) \\ &= \frac{1}{3/2} \cdot \frac{1}{5/2} \cdot \frac{1}{7/2} \cdots \\ &= \frac{2}{3} \cdot \frac{2}{5} \cdot \frac{2}{7} \cdots \end{aligned}$$

That this is the same as Euler's formula in displayed equation formula 1., when $m = \frac{1}{2}$, can be seen by

$$\begin{aligned} & \frac{1 \cdot 2^{1/2}}{1+1/2} \cdot \frac{2^{-1/2} \cdot 3^{1/2}}{2+1/2} \cdot \frac{3^{-1/2} \cdot 4^{1/2}}{3+1/2} \cdots \\ = & \frac{1}{1+1/2} \cdot \frac{2^{1/2} \cdot 2^{-1/2}}{2+1/2} \cdot \frac{3^{1/2} \cdot 3^{-1/2}}{3+1/2} \cdots \\ = & \frac{1}{3/2} \cdot \frac{1}{5/2} \cdot \frac{1}{7/2} \cdots \\ = & \frac{2}{3} \cdot \frac{2}{5} \cdot \frac{2}{7} \cdots \end{aligned}$$