Mathematics 290 Exam 5 and Final Examination Spring 2007

May 06, 2007

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Name

Technology used: ______________________________

Directions:

- **Only write on one side of each page.**
- **Use terminology correctly.**
- **Partial credit is awarded for correct approaches so justify your steps.**

Exam 5

Do Both of these "Computational" Problems

C.1. (20 points) Is $f(x) = 1+x+x^2+x^3$ in the span of $\{1+2x+9x^2+x^3,\ 9+7x+7x^3,\ 1+8x+x^2+5x^3,\ 1+8$

C.2. (10 points each) Given the linear transformation $T : P_2 \to P_2$ defined by $T(p(x)) = p(x+1)$.

(a) Find the matrix $M^T_{B,B}$ where $B = \{1, x, x^2\}$

(b) Find the algebraic and geometric multiplicities of all the eigenvalues of T.

Do Two (2) of these "In text, class or homework" problems

M.1. (20 points) Prove Theorem VRS, Vector Representation is Surjective

If $B = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ is a basis for the vector space V then The function $\rho_B : V \to \mathbf{C}^n$ given in Definition VR is a surjective linear transformation.

M.2. (20 points) Suppose that V is a vector space and $T : V \to V$ is a linear transformation. Prove that T is injective if and only if $\lambda = 0$ is not an eigenvalue of T.

M.3. (20 points) Prove Theorem FTMR, Fundamental Theorem of Matrix Representation:

Suppose that $T : U \to V$ is a linear transformation, B is a basis for U, C is a basis for V and $M^T_{B,C}$ is the matrix representation of T relative to B and C. Then, for any $\vec{u} \in U$, $\rho_C(T(u)) = M^T_{B,C}(\rho_B(\vec{u})$

Do One (1) of these "Other" problems

T.1. (20 points) The Fibonacci sequence F_n is defined by the recursion $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for each $n \geq 2$. The first few terms of the sequence are $0, 1, 1, 2, 3, 5, 8, 13, 21, \cdots$. It can be shown that the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ has the property that $[A^n]_{1,2} = F_n$. That is, for any nonnegative integer n, the entry in the first row and second column of the n^{th} power of A is the Fibonacci number F_n. Show that A is diagonalizable and use the diagonal matrix to determine a closed form for F_n. [By closed form I mean a non-recursive formula.]

T.2. (20 points) Define two vectors f, g in the vector space P_1 to be **orthogonal with respect to the coordinate basis** $B = \{1, x\}$ precisely when $\langle \rho_B(f), \rho_B(g) \rangle = 0$. [Recall that $\rho_B(f)$ is a vector in $\mathbf{C}^2$.] Find a basis for the set of all polynomials g in P_1 that are orthogonal with respect to the coordinate basis B to the polynomial $f(x) = 2x$.

Final Exam Cumulative

Do Two (2) of these "In text, class or homework" problems

CC.1. Do **one** (1) of the following:

(a) (20 points) Prove that the vector spaces M_{mn} and M_{nm} are isomorphic. Use terminology and notation correctly.

(b) (20+ points) If A is a square matrix, make a list of statements from Theorem NME, Nonsingular Matrix Equivalences. Points are taken off for incorrect statements. Extra credit for more than 10 correct statements.

(c) (20 points) Let U, V be abstract vector spaces and $T : U \to V$ a function. Show that T is a linear transformation **if and only if** for all $\vec{u}_1, \vec{u}_2 \in U$ and all scalars a, b we have $T(a\vec{u}_1 + b\vec{u}_2) = aT(\vec{u}_1) + bT(\vec{u}_2)$.[Be sure to prove **both** directions of the "if and only if".]

CC.2. (20 points) Find a basis for the kernel of the linear transformation $T : P_2 \to R^3$ given by

$$T(f) = \begin{bmatrix} f(0) \\ f'(1) \\ f(2) \end{bmatrix}.$$

CC.3. (20 points) The set $V = \text{span}\{\cos(t), \sin(t), t\cos(t), t\sin(t)\}$ is a basis for a subspace of the vector space of functions $F = \{f : \mathbf{C} \to \mathbf{C}\}$. Find the preimage of $\sin(t)$, $T^{-1}(\sin(t))$, under the linear transformation $T : V \to V$ given by $T(f) = f'$.

Do Two (2) of these "Other" problems

MM.1. (20 points) A linear transformation $T : R^{2\times 2} \to R^{2\times 2}$ is given by $T(A) = \frac{1}{2}A + \frac{1}{2}A^t$. Find all of the distinct eigenvalues of T.

MM.2. (20 points) Suppose that $T : V \to V$ is a linear transformation. Prove that $(T \circ T)(\vec{v}) = \vec{0}$ for every $v \in V$ if and only if $R(T) \subseteq K(T)$ (the range of T is a subset of the kernel of T).

MM.3. (20 points) Recall that if V is a subspace of $\mathbf{C}^n$, then the orthogonal complement of V is the set $V^\perp = \{\vec{x} \in \mathbf{C}^n : \text{ for each vector } \vec{v} \text{ in } V,\ \langle \vec{v}, \vec{x} \rangle = 0\}$. Show that $V^\perp$ is a subspace of $\mathbf{C}^n$.

MM.4. (20 points) Recall that if V is a subspace of $\mathbf{C}^n$, then the orthogonal complement of V is the set $V^\perp = \{\vec{x} \in \mathbf{C}^n : \text{ for each vector } \vec{v} \text{ in } V,\ \langle \vec{v}, \vec{x} \rangle = 0\}$. Let $B = \{\overrightarrow{v}_1, \ldots, \overrightarrow{v}_p\}$ be a basis for a subspace V of $\mathbf{C}^n$. Show that if $\overrightarrow{x} \in \mathbf{C}^n$ satisfies $\langle \overrightarrow{v}_i, \overrightarrow{x} \rangle = 0$, for all of the basis vectors $\vec{v}_i$, $i = 1, \ldots, p$ then $\overrightarrow{x} \in V^\perp$. That is, $\overrightarrow{x}$ is perpendicular to **every** vector in V and not just the vectors in the basis B.

You MUST do both of these problems.

Show your work on this page.

1. (10 points) Prove that the set $Z = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| \, 2x_1 - 4x_2 + x_3 = 0 \right\}$ is a subspace of $\mathbf{C}^3$ by applying the three-part test of Theorem TSS.

2. (10 points) Suppose that A and B are square matrices of the same size, and AB is nonsingular. Give a proof by contradiction that B is nonsingular. (Do not do this problem simply by quoting a theorem from the book.)