Mathematics 290 **Exam 3** **Spring 2007**

April 3, 2007

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Name

Technology used: ______________________

Directions:

- **Only write on one side of each page.**
- **Use terminology correctly.**
- **Partial credit is awarded for correct approaches so justify your steps.**

The Problems

Do two (2) of these computational problems

1. Show that the subset $V = \{p(x) \in P_3 :\ p(1) = p(-1),\ p(2) = p(-2)\}$ is a subspace of P_3.

   Proof:

   (a) The zero vector of P_3 is $z(x) = 0x^3 + 0x^2 + 0x + 0$.Note that $z(1) = 0 = z(-1)$ and $z(2) = 0 = z(-2)$ so $z(x) \in V$

   (b) Let $p(x)$ and $q(x)$ be in V. Then this tells us that $p(1) = p(-1),\ p(2) = p(-2)$ and $q(1) = q(-1),\ q(2) = q(-2)$

   Consider the polynomial $(p+q)(x)$: Then

$$\begin{aligned}
(p+q)(1) &= p(1) + q(1) \text{ by definition} \\
&= p(-1) + q(-1) \text{ since } p, q \in V \\
&= (p+q)(-1) \text{ by definition} \\
(p+q)(2) &= p(2) + q(2) \text{ by definition} \\
&= p(-2) + q(-2) \text{ since } p, q \in V \\
&= (p+q)(-2) \text{ by definition}
\end{aligned}$$

   Thus, $(p+q)(x)$ satisfies the definition of being in set V which tells us that V is closed under addition.

   (c) Let $p(x)$ be a vector in P_3 and α a scalar. Since $p(x)$ is in V we know that $p(1) = p(-1)$, $p(2) = p(-2)$.

   Consider the polynomial $(\alpha p)(x)$: Then

$$\begin{aligned}
(\alpha p)(1) &= \alpha p(1) \text{ by definition} \\
&= \alpha p(-1) \text{ since } p \in V \\
&= (\alpha p)(-1) \text{ by definition} \\
(\alpha p)(2) &= \alpha p(2) \text{ by definition} \\
&= \alpha p(-2) \text{ since } p \in V \\
&= (\alpha p)(-2) \text{ by definition}
\end{aligned}$$

   Thus $(\alpha p)(x)$ is a vector in V which shows that V is closed under scalar multiplication.

2. Find, with proof, a basis for the subspace $V = \{p(x) \in P_3 : \ p(1) = p(-1), \ p(2) = p(-2)\}$ of P_3.

Solution: We have

$$\begin{aligned} V &= \left\{p(x) = a_3x^3 + a_2x^2 + a_1x + a_0 : p(1) = p(-1), \ p(2) = p(-2), \ a_0, a_1, a_2, a_3 \in \mathbf{C}\right\} \\ &= \left\{ \begin{array}{c} p(x) = a_3x^3 + a_2x^2 + a_1x + a_0 : a_3 + a_2 + a_1 + a_0 = -a_3 + a_2 - a_1 + a_0 \\ \text{and } 8a_3 + 4a_2 + 2a_1 + a_0 = -8a_3 + 4a_2 - 2a_1 + a_0, \quad a_0, a_1, a_2, a_3 \in \mathbf{C} \end{array} \right\} \\ &= \left\{ \begin{array}{c} p(x) = a_3x^3 + a_2x^2 + a_1x + a_0 : 2a_3 + 2a_1 = 0 \\ \text{and } 16a_3 + 4a_1 = 0, \quad a_0, a_1, a_2, a_3 \in \mathbf{C} \end{array} \right\} \\ &= \left\{p(x) = a_3x^3 + a_2x^2 + a_1x + a_0 : a_1 = 0, a_3 = 0 \text{ and } \ a_0, a_1, a_2, a_3 \in \mathbf{C}\right\} \\ &= \left\{p(x) = a_2x^2 + a_0 : a_0, a_2 \in \mathbf{C}\right\} \\ &= \left\langle\left\{x^2, 1\right\}\right\rangle \end{aligned}$$

That $f(x) = x^2$ and $g(x) = 1$ are linearly independent can be seen by $ax^2 + b(1) = 0x^3 + 0x^2 + 0x + 0$ if and only if $a = 0$ and $b = 0$. Thus $\left\{x^2, 1\right\}$ is a basis for V and $\dim(V) = 2$.

3. Determine if the set $\left\{ \begin{bmatrix} -2 & 3 & 4 \\ -1 & 3 & -2 \end{bmatrix}, \begin{bmatrix} 4 & -2 & 2 \\ 0 & -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & -2 & -2 \\ 2 & 2 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & -2 \end{bmatrix}, \begin{bmatrix} -1 & 2 & -2 \\ 0 & -1 & -2 \end{bmatrix} \right\}$ is linearly independent in $M_{2,3}$.

Solution: A relation of linear dependence takes the form

$$a_1 \begin{bmatrix} -2 & 3 & 4 \\ -1 & 3 & -2 \end{bmatrix} + a_2 \begin{bmatrix} 4 & -2 & 2 \\ 0 & -1 & 1 \end{bmatrix} + a_3 \begin{bmatrix} -1 & -2 & -2 \\ 2 & 2 & 2 \end{bmatrix} + a_4 \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & -2 \end{bmatrix} + a_5 \begin{bmatrix} -1 & 2 & -2 \\ 0 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This can be simplified to

$$\begin{bmatrix} -2a_1 + 4a_2 - a_3 - a_4 - a_5 & 3a_1 - 2a_2 - 2a_3 + a_4 + 2a_5 & 4a_1 + 2a_2 - 2a_3 - 2a_5 \\ -a_1 + 2a_3 - a_4 & 3a_1 - a_2 + 2a_3 - a_5 & -2a_1 + a_2 + 2a_3 - 2a_4 - 2a_5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which gives us the system of equations whose augmented matrix is $\left[A|\vec{0}\right] = \begin{bmatrix} -2 & 4 & -1 & -1 & -1 & 0 \\ 3 & -2 & -2 & 1 & 2 & 0 \\ 4 & 2 & -2 & 0 & -2 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 3 & -1 & 2 & 0 & -1 & 0 \\ -2 & 1 & 2 & -2 & -2 & 0 \end{bmatrix}$,

row echelon form: $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and so the only solution to the relation of linear dependence is the trivial solution so the set of matrices is linearly independent.

Do two (2) of these problems from the text, class, old exams or homework

1. Suppose that W is a vector space with dimension 5, and U and V are subspaces of W, each of dimension 3. Prove that $U \cap V$ contains a non-zero vector. Be careful, do not assume that every basis of of U contains a vector in V.

Homework, Section PD: In class we noted that $\dim(U) + \dim(V) = \dim(U \cap V) + \dim(U + V)$. And since $U + V \subseteq W$ then $\dim(U + V) \leq 5$ giving

$$\begin{aligned} \dim(U) + \dim(V) &= \dim(U \cap V) + \dim(U + V) \\ 3 + 3 &= \dim(U \cap V) + (\text{something less than } 6) \end{aligned}$$

so $\dim(U \cap V) \geq 1$ and $U \cap V$ contains infinitely many vectors.

(a) **Alternate solution:** Let $B_1 = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ and $B_2 = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be bases for U and V, respectively and assume, for purposes of contradiction, that $U \cap V = \left\{\vec{0}\right\}$. Then starting with a relation of linear dependence we have,

$$\begin{aligned} a_1\vec{u}_1 + a_2\vec{u}_2 + a_3\vec{u}_3 + b_1\vec{v}_1 + b_2\vec{v}_2 + b_3\vec{v}_3 &= \vec{0} \\ a_1\vec{u}_1 + a_2\vec{u}_2 + a_3\vec{u}_3 &= -b_1\vec{v}_1 - b_2\vec{v}_2 - b_3\vec{v}_3 \end{aligned}$$

but the left hand side is in U and the right hand side is in V so each side individually is in $U \cap V = \left\{\vec{0}\right\}$ so we conclude that $a_1\vec{u}_1 + a_2\vec{u}_2 + a_3\vec{u}_3 = \vec{0}$ and $b_1\vec{v}_1 + b_2\vec{v}_2 + b_3\vec{v}_3 = \vec{0}$. But B_1 and B_2 are bases so they are linearly independent sets so we know that the only solutions to $a_1\vec{u}_1 + a_2\vec{u}_2 + a_3\vec{u}_3 = \vec{0}$ and $b_1\vec{v}_1 + b_2\vec{v}_2 + b_3\vec{v}_3 = \vec{0}$ are the trivial solutions which tells us that $a_1 = a_2 = a_3 = b_1 = b_2 = b_3 = 0$.This tells us the union of the sets B_1 and B_2, $B_1 \cup B_2 = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent. But this contradicts the fact that no set of 6 vectors can be linearly independent in a vector space of dimension 5.

2. Suppose that A is an invertible matrix. Prove that the matrix $\overline{(A^t)}$ is invertible and determine what that inverse is.

Solution (mimicing proofs in textbook): $\overline{(A^t)}$ is invertible if and only if there is a matrix B so that $\overline{(A^t)}B = I_n$. In this case, B is the inverse of $\overline{(A^t)}$. We show $B = \overline{(A^{-1})^t}$ works in the matrix product and hence is the desired inverse:

$$\begin{aligned} \overline{(A^t)}B &= \overline{(A^t)}\overline{(A^{-1})^t} \\ &= \overline{A^t\,(A^{-1})^t} \text{ by properties of conjugation} \\ &= \overline{(A^{-1}A)^t} \text{ by properties of transposes} \\ &= \overline{(I_n)^t} \;\; \text{since } A^{-1} \text{ is the inverse of } A \\ &= \overline{I_n} \;\; \text{since } I_n = I_n^t \\ &= I_n \;\; \text{since } I_n = \overline{I_n} \end{aligned}$$

3. Do both of the following.

(a) Prove that if V is a vector space and U and W are subspaces of V, then $U \cap W$ is a subspace of V.

Proof:

i. $\vec{0}$ is in both U and W since they are subspaces of V and every vector space contains its zero vector.

ii. Let $\vec{x}, \vec{y}$ be vectors in $U \cap W$. Since they are both in U then $\vec{x} + \vec{y} \in U$ because U is closed under addition. Similarly, since $\vec{x}, \vec{y}$ are both in W then $\vec{x} + \vec{y} \in W$ because W is closed under addition. Since $\vec{x} + \vec{y}$ is in both U and W it is in $U \cap W$ and so $U \cap W$ is closed under addition.

iii. Let $\vec{x}$ be a vector in $U \cap W$ and α a scalar. Since $\vec{x} \in U$ and U is closed under scalar multiplication, then $\alpha\vec{x} \in U$. Similarly, $\vec{x}$ is in W and W is closed under scalar multiplication so $\alpha\vec{x} \in W$. Since $\alpha\vec{x}$ is in both U and W it is in the intersection $U \cap W$. This shows that $U \cap W$ is closed under scalar multiplication.

(b) Give an example of a specific vector space V and specific subspaces U, W where $U \cup W$ is **not** a subspace of V.

Example: Let $V = \mathbf{C}^2$, $U = \left\langle \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \right\rangle$, and $W = \left\langle \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \right\rangle$. Then $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the sum of vectors in $U \cup V$ but it is not in $U \cup V$ since it is in neither U nor V.

4. Prove that if A is a square matrix where $N(A^2) = N(A^3)$, then $N(A^4) = N(A^3)$. Here $N(A^2)$ denotes the null space of A^2.

Proof: If $\vec{x} \in N(A^3)$, then $A^3\vec{x} = \vec{0}$, so $A^4\vec{x} = A\left(A^3\vec{x}\right) = A\vec{0} = \vec{0}$ and $\vec{x} \in N(A^4)$. This shows that $N(A^3) \subseteq N(A^4)$

To show $N(A^4) \subseteq N(A^3)$ we start with a vector $\vec{y} \in N(A^4)$ so that $A^4\vec{y} = \vec{0}$. Thus $A^3(A\vec{y}) = \vec{0}$ and $A\vec{y} \in N(A^3) = N(A^2)$. This tells us that $A^2(A\vec{y}) = \vec{0}$ but that is the same as $A^3\vec{y} = \vec{0}$ so $\vec{y} \in N(A^3)$. Thus $N(A^4) \subseteq N(A^3)$ and since we already proved $N(A^3) \subseteq N(A^4)$, we conclude $N(A^3) = N(A^4)$.

Do two (2) of these less familiar problems

1. Suppose that A is a square matrix and there is a vector $\vec{b}$ such that $LS\left(A, \vec{b}\right)$ has a unique solution. Prove that A is nonsingular. Note that you **do not** know that $LS\left(A, \vec{b}\right)$ has a unique solution for every $\vec{b}$. You are only told that there is a unique solution for one particular $\vec{b}$.

   Proof: The uniqueness of the solutions to $A\vec{x} = \vec{b}$ tells us that the reduced row-echelon form of $\left[A \,|\, \vec{b}\right]$ must have rank $r = n$ so that there are n columns with leading ones. The existence of a solution tells us that column $n+1$ does not have a leading one. Hence, there is a leading one in every column of the reduced row-echelon form of A which tells us that A row-reduces to I_n and so is invertible.

2. Suppose that A is an $n \times n$ matrix and B is an $n \times p$ matrix. Show that the column space of AB is contained in the column space of A.

   Solution: Let $\vec{y}$ be a vector in the column space of AB. This means there is a vector $\vec{x}$ satisfying $AB\vec{x} = \vec{y}$. Hence there is a vector $\vec{z} = B\vec{x}$ satisfying $A(B\vec{x}) = A\vec{z} = \vec{y}$ and so $\vec{y}$ is in the column space of A.

3. Let $\vec{v}$ be a particular vector in $\mathbf{C}^m$. Show that the set $V = \{\vec{w} \in \mathbf{C}^m : \vec{w} \text{ is orthogonal to } \vec{v}\} = \{\vec{w} \in \mathbf{C}^m : \langle \vec{w}, \vec{v} \rangle = 0\}$ is a subspace of $\mathbf{C}^m$. The vector space V is called the orthogonal complement of the subspace of $\mathbf{C}^m$ spanned by $\{\vec{v}\}$.

4. If $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in \mathbf{C}^3$, find a basis for the orthogonal complement of the subspace of $\mathbf{C}^3$ spanned by $\{\vec{v}\}$. [See problem 3 immediately above this problem.]