Mathematics 181 – C **Spring 2007**

Exam 2 - Key

February 27, 2007

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Name

Technology used: ____

- **Only write on one side of each page.**
- Show all of your work. Calculators may be used for numerical calculations and answer checking only.
- Be sure to include in-line citations every time you use technology and Include a careful sketch of any graph obtained by technology in solving a problem.

Do any six (6) of the following problems

datory Problem (10 points each) Do any three (3) of the following 4 integral problems.

(a) Evaluate $\int 2x \arcsin\left(x^2\right)\ dx$

Using $w = x^2$ this problem becomes $\int \arcsin(w)\ dw$.Now integration by parts gives $u = \arcsin(w)$, $dv = dw$, $v = w$, $du = \frac{1}{\sqrt{1-w^2}}dw$ so

$$\begin{aligned}\int \arcsin(w)\ dw &= w\arcsin(w) - \int w\left(1-w^2\right)^{-1/2}\ dw \\ &= w\arcsin(w) + \frac{1}{2}\int z^{-1/2}\ dz \\ &= w\arcsin(w) + z^{1/2} + C \\ &= x^2\arcsin\left(x^2\right) + \sqrt{1-x^4} + C\end{aligned}$$

where $z = 1 - w^2$ and $dz = -2w\ dw$.

(b) Use integration by parts to establish the reduction formula: $\int (\ln(x))^n\ dx = x(\ln(x))^n - n\int (\ln(x))^{n-1}\ dx$

$u = (\ln x)^n$, $dv = dx$ so $du = n(\ln x)^{n-1}\ dx$ and $v = x$ so $\int (\ln(x))^n\ dx = (\ln x)^n x - \int x\ n(\ln x)^{n-1}\ dx$

(c) Evaluate $\int 3\sec^4(3x)\ dx$

First substitute $u = 3x$ so $du = 3\ dx$ and $\int 3\sec^4(3x)\ dx = \int \sec^2(u)\sec^2(u)\ du = \int \left[\tan^2(u)+1\right]\sec^2($
$w = \tan(u)\,,\ dw = \sec^2(u)\ du)\ \int \left[w^2+1\right]\ dw = \frac{1}{3}w^3 + w + C = \frac{1}{3}\tan^3(3x) + \tan(3x) + C$

(d) Evaluate $\int 8\cos^3(2\theta)\sin(2\theta)\ d\theta$

Let $u = \cos(2\theta)$ so that $du = -2\sin(2\theta)\,d\theta$ then the integral is $\int 8\left(\frac{-1}{2}\right)u^3du = -u^4 + C = -\cos^4(2\theta) + C$

Let $u = \sin(2\theta)$ so that $du = 2\cos(2\theta)\ dy$ and $\cos^2(2\theta) = 1 - \sin^2(2\theta)$ which yields
$\int 8\cos^3(2\theta)\sin(2\theta)\ d\theta = 4\int\left[1-u^2\right]u\ du = 4\int\left(u-u^3\right)\ du = 2u^2 - u^4 + C = 2\sin^2(2\theta) - \sin^4(2\theta) + C$

Do any four (4) of the following problems.

1. (15 points) Find the length of the curve given by the equation:

$$x = \frac{y^4}{4} + \frac{1}{8y^2}, \quad 1 \le y \le 2$$

$\frac{dx}{dy} = y^3 - \frac{1}{4}y^{-3}$ so $\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + \left(y^3 - \frac{1}{4}y^{-3}\right)^2} = \sqrt{y^6 + \frac{1}{2} + \frac{1}{16}y^{-6}} = \sqrt{\left(y^3 + \frac{1}{4}y^{-3}\right)^2} = \left|y^3 + \frac{1}{4}y^{-3}\right| = y^3 + \frac{1}{4}y^{-3}$ for $1 \le y \le 2$.

So, the length is $S = \int_1^2 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_1^2 \left(y^3 + \frac{1}{4}y^{-3}\right)\, dy = \left[\frac{1}{4}y^4 - \frac{1}{8}y^{-2}\right]_1^2 = \left[\frac{16}{4} - \frac{1}{8}\left(\frac{1}{4}\right)\right] - \left[\frac{1}{4} - \frac{1}{8}\right] = \frac{123}{32} = 3.843\,75$

2. (15 points) Do one of the following

   (a) Find the area of the surface generated by revolving the curve $x = y^3/3$, $0 \le y \le 1$ about the y-axis.
   $ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \sqrt{1 + y^4} dy$ so the surface area is given by $\int x\, ds$ yielding $2\pi \int_0^1 \frac{1}{3}y^3 \sqrt{1 + y^4}\, dy = 2\pi \left(\frac{1}{9}\sqrt{2} - \frac{1}{18}\right)$ using the substitution $u = 1 + y^4$, $du = 4y^3\, dy$

   (b) Find the area of the surface generated by revolving $x = \cos(t)$, $y = 2 + \sin(t)$, $0 \le x \le 2\pi$ about the x-axis.
   $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \sqrt{[-\sin t]^2 + [\cos t]^2}\, dt = 1\, dt$ so the surface area is $2\pi \int_0^{2\pi} y\, ds = \int_0^{2\pi} [2 + \sin(t)](1)\, dt = 2\pi\, [2t - \cos(t)]_0^{2\pi} = 2\pi\, ([4\pi - 1] - [0 - 1]) = 8\pi^2$

3. (15 points) One model for the way diseases die out when properly treated assumes that the rate dy/dt at which the number of infected people changes is proportional to the the number y. That is, the number of people cured is proportional to the number y that are infected with the disease. Suppose that in any given year the number of cases can be reduced by 25%. How long will it take to eradicate the disease, that is, reduce the number of cases to less than 1? [We are given that $y(0) = 10000$ people.]

The model tells us $dy/dt = ky$ so the solution is an exponential function of the form $y(t) = y(0)\, e^{kt}$ where $y(0)$ is the initial population. Since $y(1) = \frac{3}{4}y(0)$ we have $\frac{3}{4}y(0) = y(1) = y(0)\, e^{k(1)}$ which tells us that $\frac{3}{4} = e^k$ and $k = \ln(3/4)$. We use this value of k to compute the time t when $y(t) = 10000e^{kt} = 1$.So $e^{kt} = 10^{-4}$ which implies $kt = \ln(10^{-4})$

so just over $t = \frac{\ln(10^{-4})}{\ln(3/4)} \approx 32.0$ years.

4. (15 points) Solve the separable differential equation

$$\frac{dy}{dx} = \frac{e^{2x-y}}{e^{x+y}}$$

Separating variables we have $\frac{dy}{dx} = \frac{e^{2x-y}}{e^{x+y}} = \frac{(e^{2x})(1/e^y)}{e^x e^y}$ so $\int e^{2y}\, dy = \int e^x\, dx$ giving $\frac{1}{2}e^{2y} = e^x + C$ which tells us $y = \frac{1}{2}\ln[2(e^x + C)]$

5. (15 points) A thin plate of density $\delta(x) = 4/\sqrt{x}$ covers the region between the curve $1/\sqrt{x}$ and the x-axis from $x = 1$ to $x = 16$. Find the x coordinate, $\bar{x}$, of the center of mass.

We use vertical strips. The strip located at x has center of mass $(\tilde{x}, \tilde{y}) = \left(x, \frac{1}{2}\left[1/\sqrt{x} + 0\right]\right)$ and area $\Delta A = (1/\sqrt{x})\ \Delta x$ so $\Delta m = \delta\ \Delta A = (4/\sqrt{x})\,(1/\sqrt{x})\ \Delta x = (4/x)\ \Delta x$

Thus we have total mass $M = \int_1^{16} dm = \int_1^{16} \frac{4}{x}\ dx = 4 \ln x]_1^{16} = 4\ln 16 - 4\ln 1 = 4\ln 16$

The moment $M_y = \int_1^{16} x\ dm = \int_1^{16} x\left(\frac{4}{x}\right)\ dx = \int_1^{16} 4\ dx = 4x]_1^{16} = 64 - 4 = 60.$

So the x coordinate of the center of mass is $M_y/M = (60)\,/\,(4\ln(16)) = \frac{15}{\ln(16)} \approx 5.410$