Mathematics 181 – C **Spring 2007**

Exam 3

March 27, 2007

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Name

Technology used: ____ Directions:

- Be sure to include in-line citations every time you use technology. Include a careful sketch of any graph obtained by technology in solving a problem. **Only write on one side of each page. When given a choice, specify which problem(s) you wish graded.**

The Problems

1. (10 points) Express the integrand of the following integral as a sum of partial fractions with undetermined coefficients. **Do not solve for the coefficients or evaluate the integrals.**

$$\int \frac{x^9 - 6x^5 + 7}{x\,(x+3)^4\,(x^2+4)^2\,(x^2+x+1)^2}\,dx$$

$$\int \left[\frac{A}{x} + \frac{B_1}{(x+3)^4} + \frac{B_2}{(x+3)^3} + \frac{B_3}{(x+3)^2} + \frac{B_4}{(x+3)^1} \frac{C_1x+D_1}{x^2+4} + \frac{C_2x+D_2}{(x^2+4)^2} + \frac{E_1x+F_1}{x^2+x+1} + \frac{E_2x+F_2}{(x^2+x+1)^2}\right] dx$$

2. [15 points each] Do two (2) of the following three (3) problems about integrals.

(a) Evaluate the integral

$$\int \frac{v^2 dv}{(1-v^2)^{5/2}}$$

Use $v = \sin(\theta)$ so that $dv = \cos(\theta)\,d\theta$ and, using a triangle, $\tan(\theta) = \frac{v}{\sqrt{1-v^2}}$.Then we have $\int \frac{v^2 dv}{(1-v^2)^{5/2}} = \int \frac{\sin^2(\theta)\cos(\theta)\,d\theta}{\cos^5(\theta)} = \int \tan^2(\theta)\sec^2(\theta)\,d\theta = \frac{1}{3}\tan^3(\theta) + C = \frac{1}{3}\left[\frac{v}{\sqrt{1-v^2}}\right]^3 + C.$

(b) Find the volume of the solid obtained by revolving the region bounded by $y = \frac{3}{\sqrt{3x-x^2}}$, $0.5 \le x \le 2.5$ about the x-axis.

Using the disk method the volume is $\int_{0.5}^{2.5} \pi \left(\frac{3}{\sqrt{3x-x^2}}\right)^2 dx = \int_{0.5}^{2.5} \pi \frac{9}{3x-x^2}\,dx = \pi \int_{0.5}^{2.5} \frac{9}{x(3-x)}dx = \pi \int_{0.5}^{2.5} \left[\frac{3}{x} + \frac{3}{3-x}\right] dx$ by partial fractions. This last integral is equal to $\pi \left[3\ln|x| - 3\ln|3-x|\right]_{0.5}^{2.5} = 6\pi \ln 2.5 - 6\pi \ln 0.5$

(c) Make a substitution first and then evaluate the integral

$$\int \frac{e^{4t} + 2e^{2t} - e^t}{e^{2t}+1}\,dt$$

Let $u = e^t$ so that $du = e^t\,dt$. Then, dividing after the substitution, the integral is $\int \frac{(e^{3t}+2e^t-1)}{e^{2t}+1} e^t\,dt = \int \frac{u^3+2u-1}{u^2+1} du = \int \left[u + \frac{u}{u^2+1} - \frac{1}{u^2+1}\right] du = \frac{1}{2}u^2 + \frac{1}{2}\ln|u^2+1| - \arctan(u) + C$.To finish replace all the u's with e^t.

3. [15 points] Estimate the minimum number of subintervals needed to approximate $\int_0^1 \sin(x+1)\,dx$ with an error of magnitude less than 10^{-5} using Simpson's Rule. The error bound formula is $|E_S| \le \frac{M(b-a)^5}{180n^4}$.

(a) $\left|f^{(4)}(x)\right| = |\sin(x+1)|$ which has the graph shown below for $0 \le x \le 1$. Hence we use the value $M = \sin(1) < 0.841\,5$ as a bound on $\left|f^{(4)}(x)\right|$ on that interval.

(b) Thus $|E_S| \le \frac{0.8415(1-0)^5}{180n^4} \le 10^{-5}$ if and only if $\frac{(0.8415)10^5}{180} \le n^4$ which is true when $467.5 \le n^4$ or $4.650 \le n$. Since n must be even we use $n = 6$.

4. [15 points] Do one (1) of the following two (2) problems.

(a) Determine if the following integral represents a number. If it does, find it. If it does not, explain why.

$$\int_{-2}^{3} \frac{1}{(x+1)^2}\,dx$$

This is an improper integral with an impropriety at $x = -1$. Hence, it converges if and only if both $\int_{-2}^{-1} \frac{1}{(x+1)^2}\,dx$ and $\int_{-1}^{3} \frac{1}{(x+1)^2}\,dx$ converge. Since both of these diverge the original integral diverges. Here is the work to show one diverging integral: $\int_{-1}^{3} \frac{1}{(x+1)^2}\,dx = \lim_{a\to -1^+} \int_a^3 (x+1)^{-2}\,dx = \lim_{a\to -1^+} \left[-(x+1)^{-1}\right]_a^3 = \lim_{a\to -1^+} \left[\frac{-1}{x+1}\right]_a^3 = \lim_{a\to -1^+} \left[\frac{-1}{3+1} - \frac{-1}{a+1}\right]$. Since $\lim_{a\to -1^+} \frac{-1}{a+1} = -\infty$, this integral diverges and so the original integral diverges. (A similar argument will show that $\int_{-2}^{-1} \frac{1}{(x+1)^2}\,dx$ diverges to $+\infty$.)

(b) Write the following integral (which has multiple improprieties) as the sum of improper integrals each of which has exactly one impropriety which occurs at a limit of integration. Evaluate any **one** of these integrals.

$$\int_{-2}^{\infty} \frac{1}{x(x-4)}dx$$

Since there are improprieties at 0, 4 and ∞ we must have five integrals that only have one impropriety each.
$\int_{-2}^{\infty} \frac{1}{x(x-4)}dx = \int_{-2}^{0} \frac{1}{x(x-4)}dx + \int_{0}^{3} \frac{1}{x(x-4)}dx + \int_{3}^{4} \frac{1}{x(x-4)}dx + \int_{4}^{5} \frac{1}{x(x-4)}dx + \int_{5}^{\infty} \frac{1}{x(x-4)}dx$
Using partial fractions we see that $\int_3^4 \frac{1}{x(x-4)}dx = \int_3^4 \left[\frac{-1/4}{x} + \frac{1/4}{x-4}\right]dx = \lim_{b\to 4} \left[-\frac{1}{4}\ln|x| + \frac{1}{4}\ln|x-4|\right]_3^b =$ $\lim_{b\to 4} \left[\left(-\frac{1}{4}\ln|b| + \frac{1}{4}\ln[3]\right) + \left(\frac{1}{4}\ln|b-4| - \frac{1}{4}\ln|3-4|\right)\right]$ .Since $\lim_{b\to 4}\left(\frac{1}{4}\ln|b-4|\right) = -\infty$ this integral diverges. Similar work will show that all four of the other integrals also diverge.

5. [8, 7 points] Explain whether the following infinite sequences converge or diverge and determine, with explanation, the limit of any that converge.

(a) $a_n = 3 + 2(-1)^n$
This sequence looks like $\{1, 5, 1, 5, 1, 5, \cdots\}$. Since the terms alternate between 1 and 5 the sequence can never limit to a single number. Hence this sequence diverges.

(b) $b_n = \frac{4n^4+3n}{2n^4+1000n^3} = \frac{4n^4+3n}{2n^4+1000n^3}\frac{1/n^4}{1/n^4} = \frac{4+\frac{3}{n^3}}{2+\frac{1000}{n}}$ and it is easy to see that $\lim_{n\to\infty} \frac{4+\frac{3}{n^3}}{2+\frac{1000}{n}} = \frac{4}{2} = 2$ so this sequence converges to 2.

6. [15 points] Write out the first 5 terms of the sequence of partial sums of the infinite series $\sum_{k=1}^{\infty} (-1)^n \frac{1}{n(n+1)}$.

$s_1 = -\frac{1}{2}$, $s_2 = -\frac{1}{2} + \frac{1}{6} = -\frac{1}{3}$, $s_3 = -\frac{1}{2} + \frac{1}{6} - \frac{1}{12} = -\frac{5}{12}$, $s_4 = -\frac{1}{2} + \frac{1}{6} - \frac{1}{12} + \frac{1}{20} = -\frac{11}{30}$, $-\frac{1}{2} + \frac{1}{6} - \frac{1}{12} + \frac{1}{20} - \frac{1}{30} = -\frac{2}{5}$