

# Semester Review

## The Big Picture:

**Chapter 5: Presents the basics of the theory of integration**

**Chapter 6: Standard applications of definite integrals**

**Chapter 7: How to find antiderivatives**

- (to exploit the fundamental theorem for computing definite integrals)

**Chapter 8: Sequences and Series**

- Sequences and Series are different
- How to determine convergence.
- Every power series is a function with a special domain.
- Some functions are equal to power series (their Taylor Series).

## The Medium Picture:

**Chapter 5**

**The basic theory of integration**

- Antiderivatives (From first semester calculus)
  1. Indefinite Integral notation
- Functions with Interval Domains
  1. Estimating using finite sums
  2. Sigma notation and limits of finite sums
  3. Riemann Sums and definite integrals
  4. Fundamental Theorem of Calculus
  5. Basic Substitution techniques (Rule of Thumb)
  6. Area between curves as an integral.

**Chapter 6**

**Standard Applications of Definite Integrals**

- 1. Areas between curves (from Chapter 5)
  2. Volumes of solids
    - (a) Slicing
    - (b) Rotation about an axis
  3. Arc length of curves in the plane
  4. Areas of Surfaces of Revolution
  5. Separable Differential Equations and Exponential Change

**Chapter 7**

**Methods of Integration**

- 1. Integration by parts
  2. Integrals of Trigonometric functions
  3. Computing integrals using Trigonometric Substitutions
  4. Partial Fractions for integrating **any** rational function
  5. Tables of integrals and Computer Algebra Systems
  6. Numerical Integration
  7. Improper integrals
    - Various types
    - Comparison tests to determine the **fact** of convergence.

## Chapter 8

### Infinite sequences and series

- Sequences
- Infinite Series
  1. As a limit of the **sequence of partial sums**
- Exact Sums:
  1. Geometric Series
  2. Telescoping Series
- Tests for convergence
  1. Apply to **any** series
    - (a)  $n$ th term test
    - (b) Absolute Convergence Test
      - i. Absolute convergence
      - ii. Conditional convergence
  2. Apply only to Series with Positive or non-negative terms (or the negatives of such series)
    - (a)  $P$ -Series
    - (b) Comparison Tests (direct and limit)
    - (c) Integral Test
      - i. Has a bound on error of an estimate
    - (d) Ratio and Root Tests
  3. Applies only to series whose terms alternate in sign
    - (a) Alternating Series Test
      - i. Easy bound on error of an estimate
- Power series:
  1. Every power series is a function with a special type of domain
  2. Differentiation and Integration maintain the center and radius of convergence but not convergence at endpoints
- Taylor Series and Maclaurin Series:
  1. Every function with derivatives of all orders generates a power series called the Taylor Series.
- Convergence of Taylor Series:
  1.  $R_n(x)$  determines at which values of  $x$  a function equals its Taylor Series
- Special functions and their Taylor Series
  1. Binomial Series,  $e^x$ ,  $\sin(x)$ ,  $\cos(x)$ ,  $\frac{1}{1-x}$
  2. Functions obtained from differentiating, integrating, multiplying and adding the above.

## More Detailed Outline

### Chapter 5: The fundamentals of integration

#### The basic theory of integration of Functions with Interval Domains

#### Antiderivatives (antidifferentiation)

- Reversing the process of taking derivatives.
- Harder than differentiation
- Indefinite integral notation

## Estimating using finite sums

- Areas, Distance travelled, Displacement
- Any property that can be approximated by many “smaller” and simpler structures that arise from a “nice” function.

## Sigma notation and limits of finite sums

- Changing indices in a finite sum:  $\sum_{k=1}^n f(k) = \sum_{j=4}^{n+3} f(j-3)$
- Rewriting without first few terms:  $\sum_{k=1}^n k^2 = 1 + 4 + \sum_{k=3}^n k^2 = 1 + 4 + \sum_{j=1}^{n-2} (j+2)^2$

## Riemann Sums and definite integrals:

$$\sum_{k=1}^n f(x_k^*) \Delta x_k$$

- Using sums of linear approximations over small intervals to approximate effects of functions over large intervals.
- Different Riemann sums can be obtained by varying any of the following
  1. the function  $f(x)$
  2. the interval  $[a, b]$  in the domain of  $f$
  3. the partition  $P : a = x_0 < x_1 < \dots < x_n = b$  of the interval
  4. the selection of points  $x_1^*, x_2^*, \dots, x_n^*$  where  $x_k^*$  is a point in the  $k$ 'th subinterval  $[x_{k-1}, x_k]$  of the partition.
- A definite integral is the limit, **if it exists**, as the partition norm goes to 0 of all possible Riemann sums for a function  $f$  on the interval  $[a, b]$

$$\int_a^b f(x) dx = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(x_k^*) \Delta x_k$$

1. This limit only exists if it does not matter how one partitions the interval  $[a, b]$  nor how one selects the points  $x_k^*$  in the subintervals.
2. This limit **will exist** if the function  $f$  is continuous on the interval  $[a, b]$ . (A result from advanced calculus)

## The Fundamental Theorem of Calculus

- How to compute definite integrals without using the limit of Riemann sums.
- Mean Value Theorem for Integrals and **Average Value** of a continuous function

1. Average of  $f$  on  $[a, b]$  is

$$\frac{1}{b-a} \int_a^b f(x) dx$$

2. Geometric meaning of the average value: height of rectangle over base  $a \leq x \leq b$  with same area as  $\int_a^b f(x) dx$ .

- Fundamental Theorem - Part 1. Every continuous function has an antiderivative. (Proof uses Mean Value Theorem for integrals)

$$F(x) = \int_a^x f(t) dt$$
$$F'(x) = \frac{d}{dx} \int_a^x f(t) dt = f(x)$$

- Fundamental Theorem – Part 2. Computation of definite integrals (limits of Riemann Sums) can be shortened by the use of antiderivatives (provided one can find a **nice** antiderivative for  $f$ ; the proof uses part 1 of the FTC).

$$\int_a^b f(x) dx = F(b) - F(a)$$

## Basic Integration techniques

- Substitution using  $u = g(x)$

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

- Rule of Thumb “substitute for the inside of ugliest thing” usually works for simple integrals

## Area Between Curves

- Vertical rectangles give  $\int_a^b [f(x) - g(x)] dx$
- Horizontal rectangles give  $\int_c^d [f(y) - g(y)] dy$

## Chapter 6: Standard Applications of using Riemann Sums

### Volumes of solids

Cross-sectional areas give rise to the formula

$$V = \int_a^b A(x) dx$$

- Formula applies to any solid with “nice” cross sections.
- **Special Case:** If the solid is obtained as a solid of revolution then cross sections perpendicular to the axis of revolution are particularly “nice”.
  1. Disks
  2. Washers

Nesting Cylindrical Shells gives the formula

$$V = 2\pi \int_a^b (\text{shell radius})(\text{shell height}) dt$$

- Formula applies **only** to solids of revolution and where the shells are centered on the axis of revolution.

### Arc length and Surface area:

- Use

$$\begin{aligned} ds &= \sqrt{1 + \left[\frac{dy}{dx}\right]^2} dx \\ &= \sqrt{\left[\frac{dx}{dy}\right]^2 + 1} dy \\ &= \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2} dt \\ &= \sqrt{r^2 + \left[\frac{dr}{d\theta}\right]^2} d\theta \end{aligned}$$

in the formulas:

$$\begin{aligned} \text{Length} &= \int_a^b ds \\ \text{Area} &= \int_a^b 2\pi (\text{ribbon radius}) ds \end{aligned}$$

- Many problems are ‘cooked’ so that the algebra simplifies to remove the square root.
- Arc length formula requires curve be differentiable and smooth
- Surface area formula requires the surface be obtained by rotating a curve about an axis.

## Exponential Change and Separable Differential Equations:

- Solutions to differential equations are functions that make the equation true.
  1.  $y = Ke^{3x}$  is a solution (for any choice of constant  $K$ ) to  $\frac{dy}{dx} = 3y$  because when  $y = Ke^{3x}$  we have  $\frac{dy}{dx} = 3Ke^{3x} = 3y$  showing the differential equation holds for this function  $y$ .
  2. If a differential equation also has an **initial condition**,  $y(0) = y_0$  then a solution must also make that initial condition true.
    - (a) For example,  $y = 5e^{3x}$  solves  $\frac{dy}{dx} = 3y$ , where  $y(0) = 5$  but  $y = 12e^{3x}$  does not.

- Exponential change occurs whenever a quantity changes at a rate proportional to the amount of quantity present. The model is

$$\frac{dy}{dt} = ky$$

– Examples include:

1.
    - (a) Radioactive decay
    - (b) Population growth
    - (c) Continuous interest
    - (d) Heat transfer between an object and its surroundings
- Separable differential equations are those that can be written in the form

$$h(y) \frac{dy}{dx} = g(x)$$

1. To solve, separate the variables and integrate both sides.

## Moments and Centers of Mass

- Derivation of formulas
  1. A point mass  $m_k$  located  $x_k$  units from a fulcrum has moment (tendency to rotate about the fulcrum) of  $m_k x_k$
  2. If there are finitely many masses in a system, the **System Moment** is  $\sum_{k=1}^n m_k x_k$
  3. If there is a continuous distribution of mass in a planar region.
    - (a) Cut region into strips parallel to an axis.
    - (b) Center of mass of a single narrow strip of mass  $\Delta m$  is located at a point denoted by  $(\tilde{x}, \tilde{y})$  so  $M$ , the total mass of the planar region,  $M_y$ , the moment around the  $y$  axis, and  $M_x$ , the moment around  $x$  axis are given by

$$M \approx \sum_{k=1}^n \Delta m_k$$

$$M_y \approx \frac{1}{M} \sum_{k=1}^n \tilde{x} \Delta m_k$$

$$M_x \approx \frac{1}{M} \sum_{k=1}^n \tilde{y} \Delta m_k$$

- i. So as integrals, we have

$$M = \int dm$$

$$M_y = \int \tilde{x} dm$$

$$M_x = \int \tilde{y} dm$$

- ii. Furthermore, the center of mass of the entire planar region is located at the point  $(\bar{x}, \bar{y})$  where

$$\bar{x} = \frac{M_y}{M}$$

$$\bar{y} = \frac{M_x}{M}$$

- In computations we usually use a density function  $\delta$  which gives the density in units of MASS/Area so that

$$dm = \delta dA$$

## Chapter 7: Methods of Integration

### Integration by Parts

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$
$$\int u dv = uv - \int v du$$

- Look for a product of functions,  $fg'$ , where  $g'$  has an “easy” antiderivative,  $g$ , and where the product  $gf'$  is “easier” to integrate than the original problem.

### Integrals of Trigonometric Functions

- Powers of Sine and Cosine

1. Look for an odd power of either  $\sin(x)$  or  $\cos(x)$ 
  - (a) substitute  $u$  for the other one (e.g. if  $\cos(x)$  occurs to an odd power, let  $u = \sin(x)$  so that  $du = \cos(x) dx$ )
  - (b) Use trigonometric identities to swap out even powers of the non-  $u$  trig function.
2. If both  $\sin(x)$  and  $\cos(x)$  are to even powers
  - (a) Use the half-angle trigonometric identities to reduce to an odd power

$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$$

$$\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$$

$$\sin(2x) = 2 \sin(x) \cos(x)$$

- Powers of Secant and Tangent (or Cosecant and Cotangent)

1. Look for an even power of the secant
  - (a) substitute for  $u = \tan(x)$  so  $du = \sec^2(x) dx$
  - (b) Use trigonometric identities to swap extra even powers of secant for even powers of tangent.
2. Look for an odd power of the tangent
  - (a) substitute  $u = \sec(x)$  so  $du = \sec(x) \tan(x) dx$
  - (b) Use trigonometric identities to swap extra even powers of tangent for even powers of secant
3. If secant is to an odd power and tangent is to an even power (so neither of the first two techniques work), try integration by parts with  $dv = \sec^2(x) dx$

### Trigonometric substitutions

- If  $a^2 - u^2$  occurs, try  $u = \sin(x)$  or  $u = \tanh(x)$
- If  $a^2 + u^2$  occurs, try  $u = \tan(x)$  or  $u = \sinh(x)$
- If  $u^2 - a^2$  occurs, try  $u = \sec(x)$  or  $u = \cosh(x)$

### Partial Fractions for integrating rational functions

- Only works on **proper** fractions so **divide first**.
- decompose into sums of fractions with linear, irreducible quadratic, or powers of linear or irreducible quadratic denominators
- Integrate each of the simpler fractions using other techniques

### Tables of Integrals and Computer Algebra Systems

- Be very careful when using these.
  1. Can be cumbersome to use
  2. Might have mistakes
  3. Hidden roundoff and theoretical errors in computer implementations

## Numerical Integration (Approximating definite integrals with attention to accuracy)

- Left and Right endpoint rules  $L_n$  and  $R_n$ 
  1. Simplest possible techniques to implement but not very efficient. That is, it takes a huge value of  $n$  to obtain great accuracy.
- Midpoint Rule:  $M_n$ 
  1. A bit harder to implement than  $L_n$  and  $R_n$  but much more efficient.
- Trapezoid Rule:  $T_n$  is the average of the Left and Right endpoint rules:  $T_n = \frac{L_n + R_n}{2}$ 
  1. Much more efficient than either  $L_n$  or  $R_n$
  2. On the same order of efficiency as  $M_n$ .
  3. Error Bound for  $T_n$  is:  $\left| \int_a^b f(x) dx - T_n \right| \leq \frac{(b-a)^3}{12n^2} M$
- Simpson's Rule:  $S_n = \frac{T_n + 2M_n}{3}$ 
  1. Exploits the fact that the Trapezoid error tends to be about twice the size of the Midpoint error but opposite in sign.
  2. Error Bound for  $S_n$  is:  $\left| \int_a^b f(x) dx - S_n \right| \leq \frac{(b-a)^5}{180n^4} M$

## Improper integrals

- 1. Must reduce the problem to a sum of improper integrals with **exactly one** impropriety
- 2. Types

$$\int_a^{\infty} f(x) dx$$

$$\int_{-\infty}^b f(x) dx$$

$$\int_{-\infty}^{\infty} f(x) dx$$

$$\int_a^b f(x) dx \text{ where } x = b \text{ is a vertical asymptote}$$

$$\int_a^b f(x) dx \text{ where } x = a \text{ is a vertical asymptote}$$

$$\int_a^b f(x) dx \text{ where } x = c \text{ is a vertical asymptote and } a < c < b$$

3. These integrals converge if and only if the appropriate limit exists.
  4. Direct and Limit Comparison tests
    - (a) For when exact evaluation of the integral is not possible.
    - (b) Methodology is exactly analogous to comparison tests for whether or not an infinite series converges.
- Hyperbolic Trigonometric functions

1.

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$$

$$\cosh(x) = \frac{1}{2}(e^x + e^{-x})$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}, \text{ etc.}$$

$$\cosh^2(x) - \sinh^2(x) = 1$$

2.

$$\frac{d}{dx} [\sinh(x)] = \cosh(x)$$

$$\frac{d}{dx} [\cosh(x)] = \sinh(x)$$

## Chapter 8: Sequences and Series

### Sequences

- A sequence is a function with domain the set of positive integers.
- Deduce the general term from a given sequence written in ‘dot, dot, dot’ form.
- The definition of what it means for a sequence  $a_n$  to converge:  
 $\lim_{n \rightarrow \infty} a_n = L$  means:  
Given any positive number  $\varepsilon$ , there is a number  $N$  for which whenever  $n > N$  we have

- The Nondecreasing Sequence Theorem for sequences.

A nondecreasing sequence of real numbers converges if and only if it is bounded from above. If a nondecreasing sequence converges, it converges to its least upper bound.

1. A sequence  $a_n$  is bounded above if there is a number  $M$  for which  $a_n \leq M$  for all  $n$ .
2. A sequence  $a_n$  is bounded below if there is a number  $m$  for which  $m \leq a_n$  for all  $n$ .
3. Sequences can be monotone in four ways: increasing, decreasing, nondecreasing, nonincreasing.

### Series

- Infinite Series are the discrete analogs of improper integrals of continuous functions.

$$\int_a^\infty f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx = \lim_{b \rightarrow \infty} F(x) \Big|_a^b \quad \sum_{k=1}^\infty a(k) = \lim_{n \rightarrow \infty} \sum_{k=1}^n a(k) = \lim_{n \rightarrow \infty} A(k) \Big|_1^{n+1}$$

- An infinite series converges if and only if its sequence of partial sums  $\{s_n\} = \{\sum_{k=1}^n a_k\}$  converges.
- Textbook Notation for infinite series  $\sum_{k=1}^\infty a_k$  or  $\sum_{n=1}^\infty a_n$ .
- Linearity of **convergent** series
  1. If  $\sum_{k=1}^\infty a(k)$  and  $\sum_{k=1}^\infty b(k)$  both converge then so does
  2.  $\sum_{k=1}^\infty [r a(k) + s b(k)]$  where  $r$  and  $s$  are any constants.
- If  $r$  and  $s$  are constants – neither equal to 0 then
  1. If any two of  $\sum_{k=1}^\infty a(k)$ ,  $\sum_{k=1}^\infty b(k)$ , and  $\sum_{k=1}^\infty [r a(k) + s b(k)]$  converge, then so does the third.
- Sums involving **divergent** series
  1. If  $\sum_{k=1}^\infty a(k)$  converges and  $\sum_{k=1}^\infty b(k)$  diverges then  
–  $\sum_{k=1}^\infty [r a(k) + s b(k)]$  diverges as long as  $s \neq 0$ .

### Exact Sums of Series

1. A geometric series converges if and only if  $|r| < 1$  in which case the sum is given by the formula

$$\sum_{k=0}^\infty ar^k = \frac{a}{1-r}$$

2. Telescoping series can be summed by using partial fractions to ‘telescope’ the partial sums.

### Tests that apply to any series

1.  $n$  th Term Test: An infinite series diverges if

$$\lim_{k \rightarrow \infty} a_k = \text{anything but } 0$$

- (a) Can be applied to any series
- (b) Can only inform that a series diverges – can never inform that a series converges

2. Absolute Value Series Test

- (a) If  $\sum_k^\infty |a_k|$  converges then so does  $\sum_k^\infty a_k$  and the latter’s convergence is absolute.
  - Rearrangements of absolutely convergent series do not affect either the fact of convergence or the sum.
- (b) If  $\sum_k^\infty |a_k|$  diverges and  $\sum_k^\infty a_k$  converges then the latter’s convergence is conditional.
  - A conditionally convergent series may be rearranged to converge to any number or to diverge to either plus or minus infinity.

## Tests that apply only to series with positive or non-negative terms

- $p$ - series converge if and only if  $p > 1$  (but we don't know how to find the sum)

$$\sum_{k=1}^n \frac{1}{k^p}.$$

- Integral Test

$$\sum_{k=1}^{\infty} f(k) \quad \text{and} \quad \int_1^{\infty} f(x) dx \quad \text{converge or diverge together}$$

– Applies only for a positive, decreasing continuous function  $f$

- Direct Comparison Test

1. If  $\sum_k^{\infty} c_k$  dominates  $\sum_k^{\infty} a_k$  ( $a_k \leq c_k$  for all large  $k$ ) and converges, then so does  $\sum_k^{\infty} a_k$
2.  $\sum_k^{\infty} d_k$  is dominated by  $\sum_k^{\infty} a_k$  ( $d_k \leq a_k$  for all large  $k$ ) and diverges, then so does  $\sum_k^{\infty} a_k$

- Limit Comparison Test

1. If  $\lim_{k \rightarrow \infty} \frac{a_k}{b_k} = L$ 
  - (a)  $L$  finite and non-zero, then  $\sum_k^{\infty} a_k$  and  $\sum_k^{\infty} b_k$  converge or diverge together.
  - (b)  $L = 0$  and  $\sum_k^{\infty} b_k$  converges then  $\sum_k^{\infty} a_k$  converges
  - (c)  $L = \infty$  and  $\sum_k^{\infty} b_k$  diverges then  $\sum_k^{\infty} a_k$  diverges

- Ratio Test and Root Test

1. If  $\lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k} = L$  or  $\lim_{k \rightarrow \infty} \sqrt[k]{a_k} = L$  where
  - (a)  $L < 1$  then  $\sum_k^{\infty} a_k$  converges.
  - (b)  $L > 1$  then  $\sum_k^{\infty} a_k$  diverges
  - (c)  $L = 1$  then no information

## Tests that apply only to series whose terms alternate in sign

- Alternating Series Test

1. If  $\sum_k^{\infty} a_k = \sum_k^{\infty} (-1)^k u_k$  with
  - (a)  $u_k > 0$
  - (b)  $u_k$  a decreasing sequence
  - (c)  $\lim_{k \rightarrow \infty} u_k = 0$
2. Then  $\sum_k^{\infty} a_k = \sum_k^{\infty} (-1)^k u_k$  converges.
3. Easy bound on error using an approximation:
  - (a) If  $\sum_{k=1}^{\infty} (-1)^k u_k$  converges to  $S$ , then  $\left| S - \sum_{k=1}^n (-1)^k u_k \right| < u_{n+1}$

## Power Series

- Any series in either of the forms below is a function with an interval for domain.

$$f(x) = \sum_k^{\infty} c_k x^k$$
$$f(x) = \sum_k^{\infty} c_k (x - a)^k$$

- Any power series is a function and converges on one of the following sets (which is the domain of the function. )

1. At only one point (the number  $a$ )
2. On a finite interval centered at the number  $x = a$

3. On the entire real line.

- Use Generalized Ratio or Root Tests (Apply the standard tests to the absolute value series) to detect the radius of convergence.
- Check the endpoints separately
- Power series can be differentiated and integrated term-by-term.
  1. After integrating or differentiating, the resulting series have the same Radius Of Convergence as the original series.
  2. After integrating or differentiating, the endpoints can behave differently than in the original.
- Power series can be multiplied by collecting on powers of  $x$ .
  1. The product of two power series converges only at those numbers that are in both intervals of convergence.

### Taylor Series and Maclaurin Series

- Every infinitely differentiable function  $f(x)$  gives rise to a power series centered at  $x = a$

$$\sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(a) (x-a)^k \quad (\text{Taylor Series})$$

$$\sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(0) (x-0)^k \quad (\text{Maclaurin Series})$$

- Any infinitely differentiable function  $f(x)$  satisfies Taylor's formula

$$f(x) = P_n(x) + R_n(x)$$

where  $P_n(x) = \sum_{k=0}^n \frac{1}{k!} f^{(k)}(a) (x-a)^k$  and  $R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c) (x-a)^{n+1}$  for some  $c$  between  $a$  and  $x$ .

- Hence, a Taylor series  $\sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(a) x^k$  equals its generating function  $f(x)$  if and only if

$$\lim_{n \rightarrow \infty} R_n(x) = 0$$

- We can estimate the remainder  $R_n(x)$  using

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x|^{n+1}$$

where  $M$  denotes the absolute maximum of  $|f^{(n+1)}(t)|$  on the interval between  $a$  and  $x$ .

- A few known functions and the Taylor Series they equal include:

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \quad -1 < x < 1$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad \text{for all } x$$

$$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}, \quad \text{for all } x$$

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}, \quad \text{for all } x$$

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!} x^2 + \frac{m(m-1)(m-2)}{3!} x^3 + \dots$$

The last is the binomial series and converges:

1. (a) For all  $x$  if  $m$  is an **integer** that is positive.  
(b) For  $-1 < x < 1$  if  $m \leq -1$   
(c) For  $-1 \leq x \leq 1$  if  $m > 0$  but  $m$  is **not** an integer.  
(d) For  $-1 < x \leq 1$  if  $-1 < m < 0$ .

- The Taylor series for many other functions can be computed ‘easily’ by noting that those functions are combinations of the above or the derivatives or integrals of the above.

1. **Example:**

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \quad -1 < x < 1$$

$$\begin{aligned} \frac{1}{1+x^2} &= \sum_{k=0}^{\infty} (-x^2)^k, \quad -1 < x < 1 \\ &= \sum_{k=0}^{\infty} (-1)^k x^{2k}, \quad -1 < x < 1 \end{aligned}$$

$$\begin{aligned} \text{so we have } \arctan(x) &= \int \frac{1}{1+x^2} dx \\ &= \int \sum_{k=0}^{\infty} (-1)^k x^{2k} dx \\ &= \sum_{k=0}^{\infty} \int (-1)^k x^{2k} dx \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} \quad -1 \leq x \leq 1 \end{aligned}$$