

A Gentle Introduction to Category Theory

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1 Introduction

Category theory (sometimes called “abstract nonsense”) grew out of a desire to create a general theory between different mathematical structures. Its use as a language is incredibly important in fields such as algebraic geometry and topology, as well as physics and theoretical computer science. Having a formal way of talking about the “similarities” between two structures is something that anyone who has taken an abstract algebra course has been exposed to. A homomorphism between groups for example is a special type of function that maintains the relation between the elements in the domain when they are mapped over to the codomain. In a similar fashion, category theory provides a way of comparing the structure of two different things. Instead however, of taking two groups and a map between them, category theory attempts to take the entire notion of a group and compare it to something like the notion of a ring.

Category theory has also provided a good notation for thinking about mappings. The isomorphism theorems for example can be proven simply by defining maps, but are better understood by drawing a commutative diagram. Furthermore, by understanding those relationships better, category theory can be used as a tool to map problems from one setting to another.

This paper has three main goals. The first is to provide the basic definitions of category theory such as category and functor. Second, we will show how “similar notions” such as direct product, or kernel can be unified and better understood by finding their categorical analogue. Third, we will show how category theory can be used to change between settings. Specifically, we will show how through functors we can transform topological problems into group theoretical ones by exploiting the fundamental group.

2 Categories

2.1 Basic Terminology

Before we give a formal definition of a category, we need to address a preliminary caution. A set is naively thought of as a bunch of objects, but certain collections are in a certain sense “too big” to be sets. For example, the Russel Set

$$R = \{x : x \notin x\}$$

is paradoxical and prevents the existence of a “set of all sets”. This is because for any x it will both be an element, and not be an element of R . If there were a set of all sets, we could form R by using the comprehension axiom in *ZFC*. Since we want categories to describe large and general structures such as sets, groups, rings, modules, etc. our collections will instead be called *classes*. Without going into too many details, we can think of classes as sets without certain set operations, that can be used to refer to very large collections of objects.

We are now ready to give a formal definition.

Definition. A *category* \mathcal{C} consists of three components:

1. A class of objects $\text{Ob}(\mathcal{C})$
2. A set of morphisms $\text{hom}(A, B)$ for every pair of objects in $\text{Ob}(\mathcal{C})$
3. Composition $\text{hom}(A, B) \times \text{hom}(B, C) \rightarrow \text{hom}(A, C)$, for every triple A, B, C of objects which we denote by $(f, g) \mapsto gf$. (Note, we can write $f : A \rightarrow B$ or $A \xrightarrow{f} B$ to denote $f \in \text{hom}(A, B)$).

Additionally, the above must be defined so that

- (i) the hom sets are each pairwise disjoint.
- (ii) for every object A , there is an identity morphism $1_A \in \text{hom}(A, A)$ such that

$$f1_A = f \text{ and } 1_Bf = f \text{ for all } f : A \rightarrow B$$

- (iii) Composition is associative. Namely given morphisms

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

then

$$h(gf) = (hg)f.$$

I think we are ready for some examples.

Example 1. If we let our class of objects be sets, and our morphisms the functions between them along with the usual composition of functions, we get the category **Sets**. If we take two objects A, B , then the class of functions between them is also a set (this is a basic result from set theory). So each $\text{hom}(A, B)$ is a set. The fact that the hom sets are disjoint follows readily from the definition of a function. In order for two function to be equal, they must have the same domain and range. The identity morphism is simply the identity function.

Example 2. $\mathcal{C} = \mathbf{Groups}$ is the category where $\text{ob}(\mathcal{C})$ are groups, and morphisms are homomorphisms between two groups. Again, composition is the usual functional composition. Similarly we have **Ab** where the objects are abelian groups and morphisms are homomorphisms between them.

In the two examples above, objects were either sets, or groups (which have an underlying set). Additionally, morphisms were functions. The next example will illustrate that our definition allows for very exotic categories.

Example 3. Let G be a group. Then let $\mathcal{C}(G)$ be the category with only one object denoted by $*$. Let $\text{hom}(*, *) = G$ and composition

$$\text{hom}(*, *) \times \text{hom}(*, *) \rightarrow \text{hom}(*, *);$$

or in other words, $G \times G \rightarrow G$. A group G is clearly not a function, and $*$ is a generic symbol and need not have an underlying set.

The last example was definitely contrived, but it illustrates important features about categories. If we try to describe something in the language of categories, we can not appeal to terminology about functions for example, because our morphisms can be more abstract. This brings us to our next definition, which translates the notion of an isomorphism into categorical language.

Definition. A morphism $f : A \rightarrow B$ in a category \mathcal{C} is an *equivalence* if there exists a morphism $g : B \rightarrow A$ in \mathcal{C} with

$$gf = 1_A \text{ and } fg = 1_B$$

where the morphism g is called the *inverse* of f (it is easy to show that the inverse is unique).

Before going on, we need to introduce a useful notation for categorical problems.

Definition. A *diagram* in a category \mathcal{C} is a directed multigraph, where the vertices represent objects, and the arrows represent morphisms. A path in the graph is regarded as a composition of morphisms. Additionally, a diagram *commutes* if for each pair of vertices A and B , any two paths and thus the compositions of morphisms they represent, are equal.

Example 4. The following diagram commutes if $\rho\phi = \pi$.

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ & \searrow \pi & \downarrow \rho \\ & & C \end{array}$$

2.2 The Coproduct and Product

We are now ready to explain our first example of how category theory can unify two “similar” concepts under a single categorical notion.

Recall that the *external direct sum* of a group, $G \times H$ is the group formed where the elements are tuples from the set $G \times H$ with binary operation

$$(g, h) + (g', h') = (g + g', h + h').$$

Similarly in the realm of sets, we can take two sets A, B and then define $A' = A \times \{1\}$ as well as $B' = A \times \{2\}$. Clearly $A' \cap B' = \{0\}$, and we call $A' \cup B'$ the *disjoint union* of A and B . The external direct sum and disjoint union obviously share some similarities, but this similarity can be formally captured with the categorical notion of a coproduct.

Definition. Given objects A and B in a category \mathcal{C} , their *coproduct* $A \sqcup B$ is another object in \mathcal{C} with morphisms $\alpha : A \rightarrow A \sqcup B$ and $\beta : B \rightarrow A \sqcup B$. Then for any object X in \mathcal{C} , and any pair of morphisms $f : A \rightarrow X$ and $g : B \rightarrow X$, there is a unique morphism $\theta : A \sqcup B \rightarrow X$ such that $\theta\alpha = f$ and $\theta\beta = g$. Equivalently, the following diagram must commute;

$$\begin{array}{ccc}
 & A & \\
 \alpha \swarrow & & \searrow f \\
 A \sqcup B & \overset{\theta}{\dashrightarrow} & X \\
 \beta \swarrow & & \searrow g \\
 & B &
 \end{array}$$

Now we show that disjoint union is the coproduct in **Sets**, while the external direct product is the coproduct in **Ab**.

Proposition 1. *The disjoint union of sets is a coproduct in **Sets** with injective function $\alpha : A \rightarrow A \sqcup B$ defined as $\alpha(a) = (a, 1)$ and $\beta : B \rightarrow A \sqcup B$ defined as $\beta(b) = (b, 2)$.*

Proof. If we are given two sets A and B , let $A \sqcup B = A' \cup B'$. Given a set X , and two functions $f : A \rightarrow X$ along with $g : B \rightarrow X$, we can define a function $\theta : A \sqcup B \rightarrow X$ that extends both f and g . For any $c \in A \sqcup B$, $c = (a, 1) \in A'$, or $c = (b, 2) \in B'$. We can now define $\theta((a, 1)) = f(a)$ and $\theta((b, 2)) = g(b)$. Note that θ is well defined and $\theta\alpha = f$ and $\theta\beta = g$. Now all that is left to show is that θ is unique. Assume there is another function $\psi : A \sqcup B \rightarrow X$ such that $\psi\alpha = f$ and $\psi\beta = g$, then for any $\alpha(a) \in A \sqcup B$,

$$\psi(\alpha(a)) = (\psi\alpha)(a) = f(a) = (\theta\alpha)(a) = \theta(\alpha(a)),$$

and similarly $\psi(\beta(b)) = \theta(\beta(b))$. Since together, any $\alpha(a)$ and $\beta(b)$ represent all of $A \sqcup B$, θ and ψ agree on all values. \square

We can provide a similar proof for the external direct product.

Proposition 2. *The the external direct product of groups is a coproduct in **Ab** with injective function $\alpha : A \rightarrow A \sqcup B$ defined as $\alpha(a) = (a, 0)$ and $\beta : B \rightarrow A \sqcup B$ defined as $\beta(b) = (0, b)$.*

Proof. If A and B are abelian groups, let $A \sqcup B$ be to be the external direct product $A \times B$. Now given an abelian group X along with two functions $f : A \rightarrow X$ and $g : B \rightarrow X$, define $\theta : A \times B \rightarrow X$ as $\theta((a, b)) = f(a) + g(b)$. First we must show the the diagram commutes, namely that $\theta\alpha = f$ and $\theta\beta = g$. If $a \in A$, then

$$\theta\alpha(a) = \theta((a, 0)) = f(a)$$

and similarly $\theta\beta = g$. Now we show θ is unique. Assume there is another homomorphism ψ so that $\psi((a, 0)) = f(a)$ and $\psi((0, b)) = g(b)$. Since ψ is a homomorphism we have

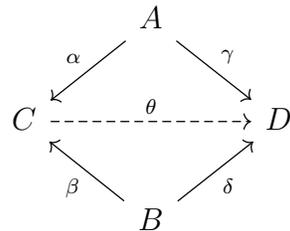
$$\begin{aligned}
 \psi((a, b)) &= \psi((a, 0) + (0, b)) \\
 &= \psi((a, 0)) + \psi((0, b)) \\
 &= f(a) + g(b)
 \end{aligned}$$

and so $\psi = \theta$. \square

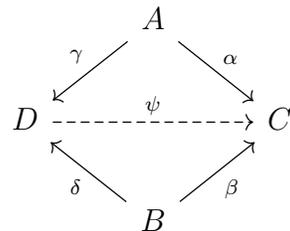
In the previous proofs we simply showed that the disjoint union and the external direct product were “a” coproduct, the next proposition says that they are in fact “the” coproducts of their respective categories.

Proposition 3. *If \mathcal{C} is a category and if A and B are objects in \mathcal{C} , then any two coproducts of A and B are equivalent.*

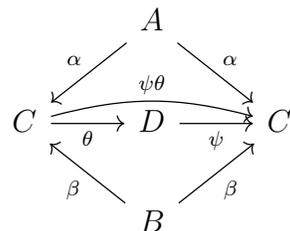
Proof. Suppose that C and D are coproducts of A and B . Along with these coproducts, we have our injective morphisms $\alpha : A \rightarrow C$, $\beta : B \rightarrow C$, $\gamma : A \rightarrow D$ and $\delta : B \rightarrow D$. Now because C is a coproduct, if we let $X = D \in \mathcal{C}$, we know that there is a unique morphism $\theta : C \rightarrow D$ such that the following diagram commutes.



We can do the same exact thing for the coproduct D , where this time we let $X = C$, which gives us a unique morphism $\psi : D \rightarrow C$ so that this diagram also commutes.



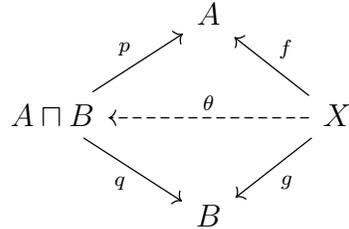
We now combine these diagrams, and claim that the result also commutes.



First observe $\psi\theta\alpha = \psi\gamma = \alpha$ and $\psi\theta\beta = \psi\delta = \beta$. But on the other hand, $1_C : C \rightarrow C$ also makes this diagram commute. Since θ and ψ are both unique, $\theta\psi$ is also unique, and so $\theta\psi = 1_C$. By drawing an analogous diagram, it would follow that $\theta\psi = 1_D$. This shows that the map $\theta : C \rightarrow D$ is an equivalence with inverse ψ . \square

Note that a coproduct need not exist, but if it does it is unique. Now we can move on to the dual concept, the product. Just as we can categorize the notion of sum, disjoint union, etc. we can categorize the notion of a cartesian product. Note the following definition is very similar to the definition of the coproduct, but this time the arrows are reversed.

Definition. Given objects A and B in a category \mathcal{C} , their *product* $A \sqcap B$ is another object in \mathcal{C} with morphisms $\alpha : A \sqcap B \rightarrow A$ and $\beta : A \sqcap B \rightarrow B$. Then for any object X in \mathcal{C} , and any pair of morphisms $f : X \rightarrow A$ and $g : X \rightarrow B$, there is a unique morphism $\theta : X \rightarrow A \sqcap B$ so that the following diagram must commute.



The cartesian product of sets is indeed the product of the **Sets**.

Proposition 4. *The cartesian product $P = A \times B$ of two sets A and B is the categorical product in sets where $p : A \times B \rightarrow A$ defined as $p((a, b)) = a$ and $q : A \times B \rightarrow B$ defined as $q((a, b)) = b$.*

Just like the coproduct, the product is also unique up to equivalence.

Proposition 5. *If A and B are objects in a category \mathcal{C} , then any two products of A and B are equivalent.*

The proofs for the two previous propositions are similar to the previous proofs about the coproduct, and so they are omitted.

Now that we have the product defined, and we know it is unique, we are prepared for a certain peculiarity about **Ab**. That is, we want to show that the coproduct is equivalent to the product in **Ab**. First recall that a group G is an *internal direct sum* of two subgroups H and K , if for each $g \in G$, there exists an $h \in H$ and $k \in K$ so that $g = h + k$, and $H \cap K = \{0\}$. If these properties hold, we write $G = H \oplus K$. We want to prove that G is the product. First however, we need to prove an equivalent notion of the internal direct sum.

Proposition 6. *If an abelian group G has subgroups H and K then the following are equivalent.*

1. $G = H \oplus K$.
2. Every $g \in G$ can be uniquely expressed in the form

$$g = h + k$$

where $h \in H$ and $k \in K$.

3. There are homomorphisms $p : G \rightarrow H$ and $q : G \rightarrow K$ and inclusions $i : H \rightarrow G$ and $j : K \rightarrow G$ such that,

$$pi = 1_H, \quad qj = 1_K, \quad pj = 0, \quad qi = 0, \quad \text{and} \quad ip + jq = 1_G.$$

The maps p and q are called **projections** while the maps i and j are called **injections**.

Proof. $1 \Rightarrow 2$. We already know that for any $g \in G$, we can write it as $g = h + k$ where $h \in H$ and $k \in K$. Suppose we can also write $g = h' + k'$. Then $h + k = h' + k'$ and $h - h' = k' - k$. Under closure of groups, $h - h' = k' - k \in H \cap K = \{0\}$. So $h = h'$ and $k = k'$

$2 \Rightarrow 3$. If we have a unique representation, then the functions p and q where $p(g) = h$ and $q(g) = k$ are well defined. The properties of 3 are then easy to confirm.

$3 \Rightarrow 1$. If we assume 3 holds, then

$$g = ip(g) + jq(g) \in H + K$$

because $H = \text{im } i$ and $K = \text{im } j$. Now we only need to show $H \cap K = \{0\}$. If $g \in H$, then $pg = p(ig) = g$ since i is merely the inclusion and $ig = 1_H$. Then if g is also in K , we have $g = pg = p(jg) = 0$ since j is merely the inclusion and $pj = 0$. So we can conclude $g = 0$, and $H \cap K = \{0\}$. \square

With this new equivalence, we can show that we have found the product in \mathbf{Ab} .

Proposition 7. *The external direct product of groups is a coproduct in \mathbf{Ab} with injective function $\alpha : A \rightarrow A \sqcup B$ defined as $\alpha(a) = (a, 0)$ and $\beta : B \rightarrow A \sqcup B$ defined as $\beta(b) = (0, b)$.*

Proof. Given two objects in our category A and B , we claim $A \sqcap B = A \oplus B$. Using our equivalence, we have projections and injections such that

$$pi = 1_A, \quad qj = 1_B, \quad pj = 0, \quad qi = 0, \quad \text{and} \quad ip + jq = 1_{A \sqcap B}$$

Now let X be another object in \mathbf{Ab} , along with given homomorphisms $f : X \rightarrow A$ and $g : X \rightarrow B$. Now define $\theta : X \rightarrow A \sqcap B$ as $\theta(x) = if(x) + jg(x)$. Now for $x \in X$, note that p is a homomorphism and so

$$p(\theta(x)) = p(if(x) + jg(x)) = pif(x) + pjg(x) = pif(x) = f(x)$$

since $pj = 0$ and $pi = 1_A$. We can similarly argue that $q\theta(x) = g(x)$. Now we only need to show that θ is unique. Note however, that $ip + jq = 1_{A \sqcap B}$ we have for another homomorphism ψ

$$\psi = (ip + jq)\psi = ip\psi + jq\psi = if + jg = \theta.$$

\square

Finally we can prove our claim that the coproduct and product are equivalent to each other in \mathbf{Ab} by forming an isomorphism between them. The fact that there is an isomorphism follows easily. Define $f : H \oplus K \rightarrow H \times K$ as $f(a) = (h, k)$. The function is well defined, because for any given a , there is a unique h and k . The fact that f is an isomorphism is then easy to show. We can now conclude that $A \sqcup B \cong A \sqcap B$ in \mathbf{Ab} (an analogous results holds in the category of modules, but does not hold in \mathbf{Groups}).

Categories have provided here an intelligible way of comparing the structures of \mathbf{Set} and \mathbf{Ab} . We were able to see how the disjoint product and external direct sum played the same categorical roles, just like Cartesian product and internal direct sum. Interestingly enough however, the product and coproduct in \mathbf{Ab} were equivalent. This result is does not hold in \mathbf{Sets} for example, because we could create a disjoint union and Cartesian product with different cardinalities (meaning no injective function could go between them).

2.3 Pullbacks and Pushouts

We have already seen that many concepts can be viewed in terms of mappings and commutative diagrams. Here we introduce two notions that will allow categorical descriptions of the kernel, and the dual concept, cokernel. First we require the following definition.

Definition. Given two morphisms $f : B \rightarrow A$ and $g : C \rightarrow A$ in a category \mathcal{C} , a *solution* is an ordered triple (D, α, β) that makes the following diagram commute:

$$\begin{array}{ccc} D & \xrightarrow{\alpha} & C \\ \downarrow \beta & & \downarrow g \\ B & \xrightarrow{f} & A \end{array}$$

We are now ready to define the pullback.

Definition. A *pullback* is a solution (D, α, β) so that for any alternative solution (X, α', β') there is a unique morphism $\theta : X \rightarrow D$ making the following diagram commute

$$\begin{array}{ccccc} X & & & & \\ & \searrow \alpha' & & & \\ & & D & \xrightarrow{\alpha} & C \\ & \searrow \theta & \downarrow \beta & & \downarrow g \\ & & B & \xrightarrow{f} & A \\ & \searrow \beta' & & & \end{array}$$

Just like the coproduct and product, pullbacks are unique as well. The proof is again, in the same fashion as the coproduct proof.

We will not tread too far deep into pullbacks, but because of its prevalence in any remotely algebraic branch of mathematics, the following is an enlightening example.

Example 5. Say we are in **Groups**. Then if $f : B \rightarrow A$ is a homomorphism and i is simply the identity restricted to 0, then the $\ker f$ is the pullback of the following diagram:

$$\begin{array}{ccc} 0 & & \\ \downarrow i & & \\ B & \xrightarrow{f} & A \end{array}$$

We claim the pullback is $K = \{(k, 0) \in B \times \{0\} : fk = 0\}$, where $\beta : K \rightarrow B$ is defined as $\beta((k, 0)) = k$ and $\alpha : K \rightarrow 0$ is defined as $\alpha((k, 0)) = 0$. K should instantly be reminiscent of the kernel. Since pullbacks are unique, all we must do is prove that K causes the diagram to commute. But this is simple, since for any $k \in K$

$$f\beta(k, 0) = f(k) = 0 = i(0) = i(g(k)) = ig(k).$$

Although we will not give a precise definition here, the dual concept **pushout**, is a pullout with arrows reversed. The **cokernel** is the solution to the same diagram as the kernel, only with the arrows once again reversed.

3 Functors

All the work done so far allows for us to understand the notion of a functor. Just as homomorphisms allow us to travel from one group, ring, or module, to another group, ring, or module, functors allow us to travel between categories.

3.1 Definition and Examples

We start by formally defining what a functor is. We present two main types of functors, beginning with the following.

Definition. Given two categories \mathcal{C} and \mathcal{D} , a *covariant functor* T is a function such that

1. if $A \in \text{Ob}(\mathcal{C})$, then $T(A) \in \text{Ob}(\mathcal{D})$;
2. if $f : A \rightarrow A'$ is an element of $\text{hom}(A, A')$ in \mathcal{C} , then $T(f) : T(A) \rightarrow T(A')$ is an element of $\text{hom}(T(A), T(A'))$ in \mathcal{D} ;
3. if $A \xrightarrow{f} A' \xrightarrow{g} A''$ is in \mathcal{C} , then $T(A) \xrightarrow{T(f)} T(A') \xrightarrow{T(g)} T(A'')$ is in \mathcal{D} and

$$T(gf) = T(g)T(f);$$

4. For all $A \in \text{Ob}(\mathcal{C})$,

$$T(1_A) = 1_{T(A)}.$$

Example 6. Let $\mathcal{C} = \mathbf{Groups}$. The *forgetful functor* $F : \mathbf{Groups} \rightarrow \mathbf{Sets}$ defined by letting $F(G)$ be the underlying set of the group G and $F(f)$ is the underlying function of a homomorphism. To be more clear, recall that a group G is actually a tuple (S, \circ) , where S is a set and \circ is the binary operation from $G \times G \rightarrow G$. $F(G) = S$. The functor gets its name because we take the group, and “forget” a certain part of it (note we could have defined analogous functors that “forget” properties for rings, modules, topological spaces, etc.)

Here is a less trivial example.

Example 7. If \mathcal{C} is a category and $A \in \text{Ob}(\mathcal{C})$, then the *hom functor* $T_A : \mathcal{C} \rightarrow \mathbf{Sets}$ is defined as

$$T_A(B) = \text{hom}(A, B) \text{ for all } B \in \text{Ob}(\mathcal{C}),$$

and if $f : B \rightarrow B'$ in \mathcal{C} , then $T_A(f) : \text{hom}(A, B) \rightarrow \text{hom}(A, B')$ is defined as

$$T_A(f) : h \mapsto fh.$$

$T_A(f)$ is called the *induced map*, and write it as f_* .

Unlike the forgetful functor, it is not obvious that that the hom functor is indeed a functor. We verify that this is the case.

Proposition 8. *The hom functor T_A , as defined above, is a functor.*

Proof. We have a category \mathcal{C} , and an object in \mathcal{C} , A . We first need to verify that $T_A(B)$ is a set, but this is true by the definition of $\text{hom}(A, B)$. Next we need to show that if f is a map between two objects in \mathcal{C} , B, B' , then $T_A(f)$ is a map between $\text{hom}(A, B)$ and $\text{hom}(A, B')$. But this makes sense because given a function between, $h \in \text{hom}(A, B)$, $T_A(f)(h) = fh$ which is a function going from $A \rightarrow B \rightarrow B$ which implies $fh \in \text{hom}(A, B')$ as needed.

Now we want to prove $T_A(fg) = T_A(f)T_A(g)$. Recall from the definition that the induced map $f_* = T_A(f)$ (this is just notationally useful). Now we want to examine the functions $(gf)_*, g_*f_*$ which both map from $\text{hom}(A, B) \rightarrow \text{hom}(A, B')$. If $h \in \text{hom}(A, B)$, then

$$(gf)_*(h) = (gf)h$$

but additionally,

$$g_*f_*(h) = g_*(fh) = g(fh) = (gf)h.$$

Now finally, if we take $1_A \in \text{hom}(A, A)$, then

$$(1_A)_*h = 1_Ah = h$$

and so $(1_A)_* = 1_{\text{hom}(A, B)}$. □

Proposition 9. *If $T : \mathcal{C} \rightarrow \mathcal{D}$ is a functor, and if $f : A \rightarrow B$ is an equivalence in \mathcal{C} , then $T(f)$ is an equivalence in \mathcal{D} . In other words, functors preserve equivalences.*

Proof. Since f is an equivalence, there exists an inverse g such that

$$gf = 1_A \text{ and } fg = 1_B.$$

Now using the property of functors

$$T(g)T(f) = T(gf) = T(1_A) = 1_{T(A)}.$$

Similarly, $T(f)T(g) = 1_{T(B)}$. So $T(f)$ is an equivalence with inverse $T(g)$. □

In addition to covariant functors, there is another type of functor which “switches” the direction of arrows.

Definition. If \mathcal{C} and \mathcal{D} are categories, then a *contravariant functor* $T : \mathcal{C} \rightarrow \mathcal{D}$ is a function such that

1. if $A \in \text{Ob}(\mathcal{C})$, then $T(A) \in \text{Ob}(\mathcal{D})$;
2. if $f : A \rightarrow B$ in \mathcal{C} , then $T(f) : T(B) \rightarrow T(A)$ in \mathcal{D} ;
3. if $A \xrightarrow{f} A' \xrightarrow{g} A''$ is in \mathcal{C} , then $T(A) \xrightarrow{T(g)} T(A') \xrightarrow{T(f)} T(A'')$ is in \mathcal{D} and

$$T(gf) = T(f)T(g);$$

4. For all $A \in \text{Ob}(\mathcal{C})$,

$$T(1_A) = 1_{T(A)}.$$

Just like there was a hom functor for covariant functors, there is also one for contravariant functors denoted T^A .

4 Viewing the Fundamental Group as a Functor

One of the pertinent questions in topology is the classification of topological spaces. Topologists capture the similarity between two spaces using continuous functions. Properties that necessarily hold when continuously mapping one space to another are said to be invariants. A common tool related to algebra is the fundamental group, which formalizes ideas regarding paths inside of a space. An equivalent topological space should have paths that behave in the same fashion. Here we give a brief exposition of the fundamental group, and relate it to the ideas discussed thus far in the paper.

4.1 Homotopies and Paths

We start by formally defining a topological space, and continuous functions between them. We then develop a more precise notion of what a path is.

Definition. Given a set X and T a family of subsets of X then T is a *topology* if

1. The empty set and X are elements of T
2. Any union of elements of T is an element of T
3. Any finite intersection of elements of T is an element of T .

The elements of T are called *open sets*.

Since we now know what an open set is, we can make sense of a continuous map.

Definition. A function $f : X \rightarrow Y$ between two topological spaces is *continuous* if for every open set U of Y , the inverse image $f^{-1}(U)$ is open in X .

Finally, we can formally capture our notion of a path.

Definition. A continuous mapping $f : [0, 1] \rightarrow X$ is called a *path* of X where $f(0)$ and $f(1)$ are the the initial and terminal point. If $f(0) = f(1)$, then the path is *closed* or a *loop*.

We now try to capture the notion of two “equivalent” paths more clearly. We start at a point x in a topological space X and compare different paths starting and ending x (in other words we are only going to think about loops). The basic intuition is that two loops are the same if you could drag one of the loops into another without ripping it apart. We begin formalizing this notion by introducing the idea of a homotopy.

Definition. Given two topological spaces X and Y along with the interval $I = [0, 1]$, two maps $f_0, f_1 : X \rightarrow Y$ are *homotopic* if there exists a continuous map $F : X \times I \rightarrow Y$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$. If F exists, it is call the *homotopy* between f_0 and f_1 .

The interval is supposed to represent how one function is being transformed into another over time. We say that two functions are *homotopic over A* if for all $a \in A$, we have $F(a, t) = f_0(a)$. Using the notion of homotopy, we can now carefully define what it means for two paths to be equivalent.

Definition. Two paths p_1, p_2 in X are *equivalent* if p_1 and p_2 are homotopic relative to $\{0, 1\}$. We then write $p_1 \sim p_2$

In other words, we allow for the paths to slowly morph into one another, but never allow for the endpoints (in this case the point we loop to) to move.

We can denote the equivalence class of equivalent paths of f as $[f]$. A proof showing that that equivalence defined above is an equivalence will be omitted (see Kosniowski for more). Now that we have classified many of the paths in a set, it is natural to ask how certain classes react with others. Could it be that traveling along one loop, and then around another loop will create the same effect as a loop in a different class? We begin by formally describing how to amend a path on top of another.

Given paths p_1 and p_2 in X , with $p_1(1) = p_2(0)$ (which will always be the case with loops), then we can create a new path $(p_1 * p_2)$ where

$$(p_1 * p_2)(t) = \begin{cases} p_1(2t) & : 0 \leq t \leq \frac{1}{2} \\ p_2(2t - 1) & : \frac{1}{2} \leq t \leq 1 \end{cases}$$

The fact that $(p_1 * p_2)$ and also that $\bar{p}_1 = p_1(1 - t)$ are paths in X is omitted, but should be intuitively clear. The fact that $(p_1 * p_2)$ is a path follows from the *gluing lemma* (see Kosniowski). Being able to add paths together, we can create an operation between equivalent classes. Although the proofs for the following will be omitted (see Kosniowski) we now have for any three loops p_1, p_2, p_3 in X and a point $x \in X$,

1. $[p_i][p_j] = [p_i * p_j]$ is well defined
2. $([p_1][p_2])[p_3] = [p_1]([p_2][p_3])$
3. $[\epsilon_x][p] = [p] = [p][\epsilon_x]$ where p is any loop and $\epsilon_x(t) = x$
4. $[p][\bar{p}] = [\epsilon_x] = [\bar{p}][p]$ for any loop p .

4.2 The Fundamental Group Induces a Functor

Here we combine the properties discussed thus far regarding equivalent paths to describe the fundamental group. We begin by restricting our talk to closed paths (or loops) only.

Definition. If $p(0) = p(1) = x$, then we say that p is *based* at x .

We can now finally introduce our group of interest.

Definition. Let X be a topological space. The set of all equivalence classes of loops based at point $x \in X$ is the *fundamental group* and we denote it $\pi(X, x)$.

Theorem 10. *The set $\pi(X, x)$ forms a group with the operation $[f][g] = [f * g]$.*

Proof. The theorem follows from observing properties introduced in the last section. Firstly, the product is well defined. The identity element is $[\epsilon_x]$, and given $[f]$, $[f]^{-1} = [\bar{f}]$ is the inverse. Additionally, associativity was also shown to hold. \square

We now know that given any $x \in X$, we can find an associated group. But what if instead of x we chose some y ? Intuitively this should not make a difference. It turns out if you can form a path from x to y inside the space, then the groups are isomorphic. The next proposition, is proven by Kosniowski.

Theorem 11. *Let $x, y \in X$. If there is a path in X from x to y , then the groups $\pi(X, x)$ and $\pi(X, y)$ are isomorphic.*

We have found a way to relate spaces to groups, let us now try and find a way to relate continuous maps to homomorphisms. If $f : X \rightarrow Y$ is a continuous map, then we have

1. If p_1 and p_2 are paths in X , then fp_1 and fp_2 are paths in Y .
2. If $p_1 \sim p_2$, then $fp_1 \sim fp_2$.
3. If p is a loop in X based at $x \in X$, then fp is a loop in Y based at $f(x)$.

So if $[p] \in \pi(X, x)$, $[fp] \in \pi(Y, p(x))$ is a well defined element. Now define $f_* : \pi(X, x) \rightarrow \pi(Y, p(x))$ as

$$f_*([p]) = [fp].$$

Proposition 12. *f_* as defined above, is a homomorphism of groups. We call it the induced homomorphism.*

Proof. Because of the machinery we have built, the proof is easy since

$$f_*([p_1][p_2]) = f_*[p_1 * p_2] = [f(p_1 * p_2)] = [fp_1 * fp_2] = [fp_1][fp_2] = f_*[p_1]f_*[p_2]$$

□

We can summarize our findings in the following way

1. For every topological space with a base point, we can find a group
2. For every continuous map between these topological spaces, we have an induced homomorphism between group
3. A composition of continuous functions induces a composition of homomorphisms
4. The identity map induces the identity homomorphism.

In other words, we have a covariant functor. In lue of this summary, we have the following theorem

Theorem 13. *Let π be a functor from the category of pointed topological spaces (morphisms are continuous maps), TP to Groups by letting $\pi(X)$ map X to its fundamental group, and mapping a continuous function f to its induced map f_* .*

Since functors preserve equivalences, the only way a topology can be the same or *homeomorphic* to another topology, is if the continuous function between them induces an isomorphism between their corresponding groups.

5 Conclusion

What we have shown is that category theory is a useful way for communicating mathematical similarities. In the first section we showed how similar notions across different disciplines can be formally described as being similar using the product and coproduct. Additionally, introducing the notion of universal mappings allows one to view concepts such as the kernel in groups, rings, modules, as the same mapping concept. Finally, category theory allows one to transport problems from category to category, as seen through the fundamental group. Although many of the concepts could have been realized without it, adding functorial language allows for a quick communication of ideas in problems to follow. Furthermore, any categorical discovery applying to functors will automatically apply to the fundamental group.

Sources

- [1] Advanced Modern Algebra, Joseph Rotman
- [2] A first course in algebraic topology, Czes Kosniowski
- [3] Algebraic Topology, Edwin H. Spanier
- [4] Basic Topology, M.A. Armstrong

Problems

Sorry I did not know if you wanted to make this public right away to I did not put solutions. Anyways I have a couple suggested problems.

- 1 (hard) Show that the hom functor, as defined above, is indeed a functor.
- 2 (medium) Draw the diagram for a coproduct and product, and give examples of each.
- 3 (hard) Prove that the coproduct is unique by assuming two different ones C and D and overlaying their commutative diagrams where $X = C$ of D .
- 4 (easy) Draw a commutative diagram that allows for a categorical notion of the kernel.